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Hecke Basis Theorems for Groups of Genus 0*

Marvin I. Knopp and John Roderick Smart****

(May 1, 1970)

A Hecke-type basis theorem is established for the cusp forms of negative even integral degree (multiplier system 1) on the class of Hecke groups. Hecke established the result for the classical modular group, which is the first of the Hecke groups. A second result is a parametrization theorem for entire automorphic forms of negative real degree (with arbitrary multiplier systems) on certain discrete groups of real linear fractional transformations of genus zero.

Key words: Eisenstein series; Hecke group; H-group; modular form; parametrization of cusp forms.

1. Introduction

In [11, Theorem 2][1] Rademacher and Zuckerman establish a "parametrization theorem" for modular forms of all real degrees and as a consequence give an explicit parametric representation for all multiplier systems connected with the modular group $\Gamma(1)$. In a different (but related) direction, Hecke [1, p. 3] earlier established the existence of a particularly useful special type of basis for the cusp forms of negative even integral degree with multiplier system identically 1 on $\Gamma(1)$; this basis is given in terms of the well-known modular form

$$\Delta(z)=e^{2\pi iz}\prod_{m=1}^{\infty}(1-e^{2\pi imz})^{24}, \tag{1.1}$$

and the "Eisenstein series"

$$E_k(z)=\sum(cz+d)^{-k}, \tag{1.2}$$

where k is an even integer ≥ 4 and the summation is on all pairs of relatively prime rational integers c, d. $\Delta(z)$ is a *cusp form* of degree -12 and multiplier system identically 1 on $\Gamma(1)$, whereas $E_k(z)$ is an *entire modular form* of degree $-k$ and multiplier system 1, but not a cusp form since $E_k(z)\to 2$ as $z\to i\infty$. It is clear from the representation (1.1) that $\Delta(z)$ has no zeros in $\mathscr{H}$, the upper half-plane.

Theorem 1 of this article is a Hecke-type basis theorem for the cusp forms of negative even integral degree with multiplier system 1 on an infinite class of discrete groups $\Gamma(\lambda_n)$, n an integer ≥ 3, of linear fractional transformations preserving $\mathscr{H}$. $\Gamma(\lambda_n)$, called the nth *Hecke group* (or *the Hecke group of parameter* n), is the group generated by the two matrices

$$S_n=\begin{pmatrix}1 & \lambda_n\\ 0 & 1\end{pmatrix},\qquad T=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

with $\lambda_n=2\cos(\pi/n)$. Here we have adopted the convention, to which we adhere throughout, that the matrix $M=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is to be identified with the linear fractional transformation

$$z\to(az+b)/(cz+d)=Mz.$$

Note that $-M$ is then identified with the same linear fractional transformation. In particular the matrix $-I$ is the identity when thought of as a linear fractional transformation. $\Gamma(1)$, the modular group, is the Hecke group of parameter 3; that is $\Gamma(1)=\Gamma(\lambda_3)$, since $\Gamma(1)$ is generated by

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[1] Figures in brackets indicate the literature references at the end of this paper.

$$U=S_3=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \quad \text{and} \quad T=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

Our second principal result, given in section 7 and summarized in table 2, is a parametrization theorem for entire automorphic forms of negative *real* degree on certain discrete groups of linear fractional transformations acting on $\mathscr{H}$, of genus zero. Among the groups included are $\Gamma(1)$; Γ_ϑ, the group connected with the well-known modular form $\vartheta(z)=\vartheta_3(0|z)$ and generated by

$$U^2=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix} \quad \text{and} \quad T=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix};$$

$\Gamma(2)$ and $\Gamma(3)$, the principal congruence subgroups of levels 2 and 3 (For n a positive integer $\Gamma(n)$ is the normal subgroup of $\Gamma(1)$ defined by the congruence condition

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\equiv \pm \mathrm{I} \pmod{n}.);$$

$\Gamma(\lambda_4)=\Gamma(\sqrt{2})$ and $\Gamma(\lambda_6)=\Gamma(\sqrt{3})$. For these groups the entire forms are parametrized and the multiplier systems are explicitly determined. Other groups of genus zero are also included in table 2, but the results are neither as complete nor as explicit as in the cases listed above. It is of interest to note that the multiplier systems connceted with entire forms of negative degree on a discrete group Γ include *all* the multiplier systems connected with Γ. Thus in those cases where we explicitly calculate the multiplier systems for entire forms, we have in fact determined all multiplier systems.

The parametrization theorem of Rademacher and Zuckerman expresses all modular forms (of all degrees) in terms of a small number (four) of familiar modular functions and forms. In contrast, our results of section 7 express all entire forms of a *fixed* degree $-r$ in terms of a finite set of entire forms of degree $-r$; the number of functions in the set equals the dimension of the space of entire forms of degree $-r$ and therefore increases with r. Essentially the same types of functions occur here as in theorem 1 and we would therefore be justified in referring to the results of section 7 as "Hecke basis theorems", generalized to include the treatment of entire forms of *arbitrary* negative real degree, with *arbitrary* multiplier systems.

The function $\Delta(z)$ defined by (1.1) plays an important role in Hecke's original basis theorem as it has no zeros in $\mathscr{H}$ and a zero of order 1 (in the uniformizing variable $e^{2\pi iz}$) at $i\infty$. To obtain our basis theorem for the Hecke groups $\Gamma(\lambda_n)$, we need a cusp form on $\Gamma(\lambda_n)$ with similar properties. In section 3 we show that, for an arbitrary H-group Γ of genus 0, there exists a cusp form $\Delta(\Gamma, r, 1; z)$ of negative even integral degree $-r$ and multiplier system 1 with no zeros in $\mathscr{H}$. An appropriate power of $\Delta(\Gamma, r, 1; z)$ will then be a cusp form with a zero of order exactly 1 at $i\infty$. In the case of the group $\Gamma(\lambda_n)$ we denote this cusp form $\Delta_1(z; n)=\Delta_1(z)$. As might be expected $\Delta_1(z; 3)=\Delta(z)$, the function defined by (1.1).

An explicit statement of theorem 1 requires the introduction of Eisenstein series in a more general sense than that of (1.2). In fact Eisenstein series can be defined for all H-groups and in more general situations as well. In section 2 we give the definition of the Eisenstein series

$$E_r(\Gamma, v; z),$$

an entire automorphic form of real degree $-r\leq -2$, with multiplier system v, on an arbitrary H-group Γ. In the case $r=2$ convergence difficulties present themselves and the Eisenstein series $E_2(\Gamma, v; z)$ can be defined only if $v\equiv 1$. For notational convenience we also define $E_0(\Gamma, v; z)=1$. With k an even integer ≥ 4 consider the functions

$$B_l=E_{k-4nl/(n-2)}(\Gamma(\lambda_n), v_1^{-1}; z)\cdot \Delta_1^l(z), \quad 0<l<(k-2)(n-2)/4n, \tag{1.3}$$

where l is understood to be an integer and v_1 is the multiplier system of the cusp form $\Delta_1(z)$.

The upper inequality is imposed upon l to insure that $k-4nl/(n-2)>2$. We also include the function

$$B_l=E_2(\Gamma(\lambda_n), v_1^{-l}; z)\cdot\Delta_1^l(z), \qquad l=(k-2)(n-2)/4n, \tag{1.4}$$

if $4n|(k-2)(n-2)$ and $v_1^l\not\equiv 1$. Finally we consider the function

$$B_l=E_0(\Gamma(\lambda_n), v_1^{-l}; z)\cdot\Delta_1^l(z), \qquad l=k(n-2)/4n. \tag{1.5}$$

if $4n|k(n-2)$ and $v_1^l\equiv 1$. The functions B_l of (1.3), (1.4), and (1.5) are all cusp forms of degree $-k$ with multiplier system 1 on $\Gamma(\lambda_n)$, and furthermore they are linearly independent since B_l has a zero at $i\infty$ of order exactly l. The question of when they form a basis for the vector space of such cusp forms is dealt with in theorem 1 which follows.

THEOREM 1: *Let* k *be an even integer* ≥ 4. *With* n *an integer* ≥ 3, *write* $k=4ns+2p$, s *an integer* ≥ 0 *and* $0\leq p<2n$. *Then the functions* B_l *form a basis for the complex vector space of cusp forms of degree* $-k$ *with multiplier system* 1 *on the group* $\Gamma(\lambda_n)$ *except in the cases*

$$\frac{n}{2}+1<p<n, \qquad p \text{ even} \tag{1.6}$$

and

$$\frac{3n}{2}+1<p<2n, \qquad p \text{ even}. \tag{1.7}$$

In the exceptional cases (1.6) *and* (1.7) *the number of functions* B_l *is* 1 *less than the dimension of the space of cusp forms.*

REMARKS: 1. B_l has a zero of order exactly l (in the uniformizing variable $e^{2\pi iz/\lambda_n}$) at $i\infty$. This property of the set of functions B_l is the property that justifies our calling the set of B_l's a "Hecke basis" in those cases when it indeed is a basis.

2. In the exceptional cases (1.6) and (1.7) the "missing function" is the one with highest possible order zero at $i\infty$.

3. If $n=3$, 4, or 6 there are no integers p satisfying (1.6) or (1.7). Thus in these cases a Hecke basis is always achieved. The groups in question are $\Gamma(1)$, $\Gamma(\sqrt{2})$, and $\Gamma(\sqrt{3})$, respectively. In particular, then, Hecke's original basis theorem is contained in theorem 1.

The proof of theorem 1 is given in section 5, while section 6 is devoted to a short discussion of the missing function in the exceptional cases (1.6) and (1.7). In section 4 a Hecke basis theorem is given in the case of a restricted class of H-groups for which a very simple proof suffices. Among the groups included there are $\Gamma(1)$ and $\Gamma(\sqrt{2})$ again.

2. Preliminary Definitions and Notation

An *H-group* Γ is a discrete subgroup of $SL(2, R)$ (thought of as linear fractional transformations), where R is the real numbers, such that

(i) Γ is finitely generated,

(ii) Γ contains translations,

and

(iii) every point of the real axis is a limit point for Γ.

To put this another way, an H-group is a finitely generated Fuchsian group of the first kind, with translations, acting on $\mathscr{H}$. Such a group has a presentation of the form [7, p. 173]

$$\Big\{A_1, B_1, \ldots, A_g, B_g, E_1, \ldots, E_e, P_1, \ldots, P_t, -I;\ (-I)^2=I,$$
$$E_1^{l_1}=\ldots=E_e^{l_e}=-I, \qquad \prod_{i=1}^{g}\gamma_i\prod_{m=1}^{e}E_m\prod_{h=1}^{t}P_h=(-I)^{e+t}\Big\}, \tag{2.1}$$

where l_m, $1\leq m\leq e$, is an integer ≥ 2, $\gamma_i=A_iB_iA_i^{-1}B_i^{-1}$, $1\leq i\leq g$, the A_i and B_i are hyperbolic

matrices, the E_m are elliptic matrices, the P_h are parabolic matrices, and I is the 2×2 identity matrix. The numbering is so arranged that $P_t=\begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix}$, $\lambda>0$, a translation by the real number λ. When the group Γ has the presentation (2.1) we say that Γ is of *signature* $(g, n; l_1, \ldots, l_e)$, where $n=e+t$. The integer g is called the *genus* of Γ. In particular, the modular group $\Gamma(1)$ is of signature (0, 3; 2, 3), with generating matrices

$$E_1=-T=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \qquad E_2=\begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix}, \qquad P_1=U=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}.$$

In this article we shall be concerned only with H-groups Γ of genus $g=0$, so we may assume that Γ has the presentation

$$\left\{E_1, \ldots, E_e, P_1, \ldots, P_t; E_1^{l_1}=\ldots=E_e^{l_e}=-I, \prod_{m=1}^{e} E_m \prod_{h=1}^{t} P_h=(-I)^{e+t}\right\}. \tag{2.2}$$

Let r be a real number and let Γ be an H-group. We want to consider functions $F(z)$, meromorphic in $\mathscr{H}$, such that

$$F(Vz)=v(V)(cz+d)^rF(z) \tag{2.3}$$

for every $V=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma$ and z in $\mathscr{H}$. Here $v(V)$ is a complex number of absolute value 1 independent of z. In order to fix the branch of $(cz+d)^r$ when r is not an integer, we define, for any complex number τ and real number s,

$$\tau^s=|\tau|^s\cdot\exp(i\,s\,arg\,\tau), \qquad -\pi<arg\,\tau\leq\pi. \tag{2.4}$$

When such a function $F(z)\not\equiv 0$ exists, it follows that

$$v(V_1V_2)(c_3z+d_3)^r=v(V_1)v(V_2)(c_1V_2z+d_1)^r(c_2z+d_2)^r, \tag{2.5}$$

where $V_i=\begin{pmatrix}* & *\\ c_i & d_i\end{pmatrix}\in\Gamma$ for $i=1, 2$ and $V_1V_2=\begin{pmatrix}* & *\\ c_3 & d_3\end{pmatrix}$. A function v from Γ to the complex numbers of absolute value 1 and satisfying the *consistency condition* (2.5) is called a *multiplier system for* Γ *of degree* $-r$.

Each parabolic generator P_h of Γ has a single fixed point and it is real. This point x_h is called a *parabolic cusp;* the stabilizer of x_h in Γ is the cyclic subgroup of Γ generated by P_h, $1\leq h\leq t$. From this latter fact, equation (2.3), and the fact that $F(z)$ is meromorphic in $\mathscr{H}$ it follows [4, pp. 272–273] that $F(z)$ has an expansion "at x_h" of the form

$$F(z)=(z-x_h)^{-r}\sum_{n=-\infty}^{\infty} b_n(h)e^{2\pi i(n+\kappa_h)(A_hz)/\lambda_h}, \tag{2.6}$$

where $\lambda_h>0$, A_h is a real linear fractional transformation such that $A_h(x_h)=\infty$, and κ_h is defined by $v(P_h)=e^{2\pi i\kappa_h}$, $0\leq\kappa_h<1$.

Suppose that $F(z)$ is meromorphic in $\mathscr{H}$, satisfies (2.3), and for each h, $1\leq h\leq t$, only finitely many terms such that $n<0$ appear in (2.6). Then we say that $F(z)$ is an *automorphic form of degree* $-r$ *with respect to the group* Γ *and the multiplier system* v. If $F(z)$ is holomorphic in $\mathscr{H}$ and for each h only terms such that $n+\kappa_h\geq 0$ appear in (2.6) we say that $F(z)$ is an *entire* automorphic form. If $F(z)$ is an entire form such that only terms with $n+\kappa_h>0$ appear in (2.6) we say that $F(z)$ is a *cusp form.* The complex vector space of automorphic forms of degree $-r$ with multiplier system v on Γ is denoted $\{\Gamma, -r, v\}$. The space of entire forms is denoted $C^+(\Gamma, -r, v)$ and that of cusp forms $C^0(\Gamma, -r, v)$. A *modular form* is simply an automorphic form with respect to the modular group $\Gamma(1)$.

Besides the expansions (2.6) at the parabolic cusps (and of course the usual Taylor or Laurent expansions at points of $\mathscr{H}$), an automorphic form $F(z)$ in $\{\Gamma, -r, v\}$ has special expansions at the fixed points ω_m in $\mathscr{H}$ of the elliptic transformations E_m, $1 \leq m \leq e$[7, pp. 180–181]:

$$F(z) = (z-\bar{\omega}_m)^{-r} \sum_{n=\nu_m}^{\infty} b_n(\omega_m) u^{n+a_m/l_m}, \tag{2.7}$$

where l_m is the order of E_m in the sense of (2.2), $u = \{(z-\omega_m)/(z-\bar{\omega}_m)\}^{l_m}$, and

$$v(E_m) = e^{\pi i r/l_m + 2\pi i a_m/l_m}, \qquad 0 \leq a_m \leq l_m - 1, \tag{2.8}$$

with a_m an integer. Now (2.6) and (2.7) enable us to define the *order* of $F(z)$ at all points (including the parabolic cusps) of a fundamental region $\mathscr{R}$ for Γ in $\mathscr{H}$. The order of $F(z)$ at points of $\mathscr{H}$ not fixed by elliptic transformations in Γ is defined in the usual way. $\mathscr{R}$ can be so constructed that the only points in $\mathscr{R}$ fixed by elliptic transformations in Γ are the points ω_m fixed by the transformations E_m, $1 \leq m \leq e$. At ω_m the order of $F(z)$ is defined to be $\nu_m + a_m/l_m$. Finally suppose that in (2.6) the nonzero $b_n(h)$ with smallest subscript occurs for $n = n_h$. Then we define the order of $F(z)$ at the parabolic cusp x_h to be $n_h + \kappa_h$. With these definitions the sum of orders of $F(z) \in \{\Gamma, -r, v\}$ at points in $\mathscr{R}$ is $r\left(g - 1 + \frac{q}{2}\right)$, where

$$q = t + \sum_{m=1}^{e} \left(1 - \frac{1}{l_m}\right) \tag{2.9}$$

and g is the genus of Γ [6, p.191]. This result can be proved, for example, by integrating $F'(z)/F(z)$ over the boundary of $\mathscr{R}$. In particular, if $F(z) \in C^+(\Gamma, -r, v)$, then the total number N of *zeros* of $F(z)$ *in* $\mathscr{R}$ is given by the formula

$$N = r\left(g - 1 + \frac{q}{2}\right). \tag{2.10}$$

Note that N need not be an integer.

We next turn to the promised definition of the Eisenstein series $E_r(\Gamma, v; z)$. Suppose, as before, that in the presentation (2.2) of Γ the numbering is arranged so that $P_t = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, $\lambda > 0$. Let v be a multiplier system for Γ of degree $-r$ such that, in addition, $v(P_t) = 1$; this is equivalent to assuming that $\kappa_t = 0$. If $r \geq 2$, then put

$$E_r(\Gamma, v; z) = \sum \bar{v}(V)(cz+d)^{-r}, \tag{2.11}$$

where the summation is over all elements $V = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$ with distinct lower rows. It can be shown that since $v(P_t) = 1$, $v(V)$ is independent of the upper row of V and therefore (2.11) is unaffected by the choice of V with lower row c, d. If $r > 2$, then the series in (2.11) converges absolutely. From the absolute convergence and (2.5) it is easy to deduce that $E_r(\Gamma, v; z) \in C^+(\Gamma, -r, v)$. If $r = 2$ then there are convergence difficulties which have been overcome by Petersson. In this case the series (2.11), suitably interpreted, is in $C^+(\Gamma, -2, v)$, provided only that $v \neq 1$ on Γ.

3. The Existence of $\Delta(\Gamma, r, v; z)$ When $g = 0$

We begin with:

THEOREM 2: *Suppose* Γ *is an* H*-group of genus* 0 *and* r *is a real number* > 2. *Then there exists a multiplier system* v *corresponding to the group* Γ and the degree $-$r *with the following property:*

$$\begin{aligned} &\textit{If dim } C^0(\Gamma, -r, v) > 0, \textit{ then there is a cusp form} \\ &\Delta(\Gamma, r, v; z) \in C^0(\Gamma, -r, v) \textit{ which has no zeros in } \mathscr{H}. \end{aligned} \tag{3.1}$$

PROOF: As before, we write

$$v(E_m)=\exp(\pi i r/l_m+2\pi i a_m/l_m),\qquad 0\leq a_m\leq l_m-1;\ 1\leq m\leq e,$$
$$v(P_h)=\exp(2\pi i\kappa_h),\qquad 0\leq\kappa_h<1;\ 1\leq h\leq t, \tag{3.2}$$

with a_m an integer. Furthermore we put

$$\vartheta_h=\begin{cases}1 & \text{if } \kappa_h=0\\ 0 & \text{if } \kappa_h>0\end{cases},\qquad 1\leq h\leq t. \tag{3.3}$$

Put $\mu=\dim C^0(\Gamma,-r,v)$; then by Petersson's Riemann-Roch Theorem [6, p. 194],

$$\mu=-\sum_{h=1}^{t}(\vartheta_h+\kappa_h)+r\left(\frac{q}{2}-1+g\right)-\sum_{m=1}^{e}a_m/l_m+1, \tag{3.4}$$

where q is defined by (2.9).

We want to show the existence of a multiplier system v corresponding to Γ and the degree $-r$ such that v has the property (3.1). It is not too difficult to show [4, p. 273] that if $a_m>0$ then *any* element of $C^0(\Gamma,-r,v)$ must have a zero at the fixed point of E_m. Since such a fixed point lies in $\mathscr{H}$ any multiplier system v satisfying (3.1) must be such that $a_m=0$ for $1\leq m\leq e$. Accordingly we assume $a_m=0$ for $1\leq m\leq e$. Petersson [6, pp. 534–535] has proved that a necessary and sufficient condition for the existence of a multiplier system v corresponding to Γ and the degree $-r$, satisfying (3.2), is that the a_m and κ_h satisfy the congruence

$$\sum_{h=1}^{t}\kappa_h+\sum_{m=1}^{e}a_m/l_m\equiv r\left(\frac{q}{2}-1+g\right)\pmod 1. \tag{3.5}$$

Since $g=0$ here and we have assumed $a_m=0$ for $1\leq m\leq e$, this congruence takes the form

$$\sum_{h=1}^{t}\kappa_h\equiv r\left(\frac{q}{2}-1\right)\pmod 1. \tag{3.5'}$$

Because $t\geq 1$ we can obviously choose the κ_h, $1\leq h\leq t$, in such a way that (3.5′) is satisfied. Alternatively, we may avoid Petersson's condition if we so desire, by simply defining v on the E_m and P_h by means of (3.2), with all $a_m=0$, and the κ_h chosen to be consistent with the relation

$$\prod_{m=1}^{e}E_m\prod_{h=1}^{t}P_h=(-I)^{e+t}$$

and the consistency condition (2.5). It is not hard to check that this procedure gives rise to a bona fide multiplier system, since $g=0$ (and thus Γ is generated by the E_m and the P_h). In fact we may *arbitrarily* choose $\kappa_1,\ \ldots,\ \kappa_{t-1}$; then κ_t is determined by the above relation. Thus if $t=1$ there is a uniquely determined multiplier system v corresponding to Γ and the degree $-r$ with $a_m=0$ for $1\leq m\leq e$ and if $t>1$ there is an infinite class of such multiplier systems parametrized by the real parameters $\kappa_1,\ \ldots,\ \kappa_{t-1}$.

We shall complete the proof by showing that *any* multiplier system v with $a_m=0$ for $1\leq m\leq e$ has the property (3.1). Thus assume $a_m=0$ for $1\leq m\leq e$, $g=0$, and $\mu>0$. Then (3.4) takes the form

$$\mu=-\sum_{h=1}^{t}(\vartheta_h+\kappa_h)+r\left(\frac{q}{2}-1\right)+1. \tag{3.6}$$

Now any function in $C^0(\Gamma, -r, v)$ has a zero at the parabolic cusp corresponding to P_h of order at least $\vartheta_h + \kappa_h$, $1 \leq h \leq t$. Thus if we succeed in constructing $\Delta = \Delta(\Gamma, r, v; z) \epsilon C^0(\Gamma, -r, v)$ such that Δ has a zero at $i\infty$ of order $N - \sum_{h=1}^{t-1} (\vartheta_h + \kappa_h)$, with N as in (2.10), then Δ will have no zeros in $\mathscr{H}$ and the proof will be complete. Let $\varphi_1, \ldots, \varphi_\mu$ be a basis for $C^0(\Gamma, -r, v)$ and suppose φ_j has the expansion at $i\infty$

$$\varphi_j(z) = \sum_{n \geq \vartheta_t} b_n(j) e^{2\pi i (n+\kappa_t) z/\lambda}, \quad 1 \leq j \leq \mu.$$

Form the linear combination

$$\Delta(z) = \sum_{j=1}^{\mu} \zeta_j \varphi_j(z) = \sum_{n \geq \vartheta_t} \left(\sum_{j=1}^{\mu} \zeta_j b_n(j) \right) e^{2\pi i (n+\kappa_t) z/\lambda},$$

and note that $N - \sum_{h=1}^{t-1} (\vartheta_h + \kappa_h) = \mu - 1 + \vartheta_t + \kappa_t$. It suffices to solve the homogeneous equation

$$\sum_{j=1}^{\mu} \zeta_j b_n(j) = 0, \qquad \vartheta_t \leq n \leq \mu + \vartheta_t - 2$$

with $\zeta_1, \ldots, \zeta_\mu$ not all zero. But this is a system of $\mu - 1$ equations in μ unknowns and thus has a nontrivial solution. The proof is complete.

For each H-group Γ of genus zero we want to demonstrate the existence of at least one cusp form $\Delta(\Gamma, r, v; z)$ that is zero-free in $\mathscr{H}$. By theorem 2 it suffices to show that for each such group Γ there exists $r > 2$ and a corresponding multiplier system v such that $\dim C^0(\Gamma, -r, v) > 0$.

LEMMA 3: *Suppose* Γ *is an* H*-group of genus zero. If* $e = 0$, *let* $r = 6$. *If* $e = 1$, $l_1 = 2$, *and* $t = 2$, *choose* $r = 8$. *Otherwise put* $r = 2 \cdot lcm\ (l_1, \ldots, l_e)$. *In every case let* $v \equiv 1$ *on* Γ. *Then dim* $C^0(\Gamma, -r, 1) > 0$.

PROOF: With r chosen as in the statement of the lemma we have $2l_m | r$ for $1 \leq m \leq e$. Thus $a_m = 0$ for $1 \leq m \leq e$. Likewise $\kappa_h = 0$ for $1 \leq h \leq t$. Then by (3.4)

$$\mu = \dim C^0(\Gamma, -r, 1) = -t + r\left(\frac{q}{2} - 1\right) + 1.$$

Note that in every case $r \geq 4$. Thus if $q \geq 2$ it follows that $\mu \geq t + 2 \sum_{m=1}^{e} \left(1 - \frac{1}{l_m}\right) - 3$. In particular if $t \geq 4$ or $t = 3$ and $e \geq 1$, then $\mu > 0$. If $t = 3$ and $e = 0$, we choose $r = 6$ and find that $\mu = 1$.

Suppose $t = 2$. Then $\mu = \frac{r}{2} \sum_{m=1}^{e} \left(1 - \frac{1}{l_m}\right) - 1$. Since $l_m \geq 2$, $\mu \geq \frac{re}{4} - 1$. With $e \geq 2$, $\mu > 0$ because $r \geq 4$. If $e = 1$ and $l_1 > 2$, then again $\mu > 0$. If $e = 1$ and $l_1 = 2$, choose $r = 8$ and $\mu = 1$.

Suppose $t = 1$. We apply the well-known inequality due to Siegel [12, theorem 5, p. 715]

$$2g - 2 + t + \sum_{m=1}^{e} \left(1 - \frac{1}{l_m}\right) \geq 1/42, \tag{3.7}$$

where Γ is an H-group of genus g. In our application here $g = 0$ and $t = 1$, so that (3.7) becomes

$$\sum_{m=1}^{e} \left(1 - \frac{1}{l_m}\right) - 1 \geq 1/42. \tag{3.8}$$

From (3.8) we deduce immediately that $e \geq 2$. If $e \geq 3$, then

$$\mu=\frac{r}{2}\left\{\sum_{m=1}^{e}\left(1-\frac{1}{l_m}\right)-1\right\}\geq\frac{r}{2}\left(\frac{e}{2}-1\right)>0.$$

If $e=2$, then by (3.8) either $l_1\geq 3$ or $l_2\geq 3$. Thus, once more,

$$\mu=\frac{r}{2}\left(1-\frac{1}{l_1}-\frac{1}{l_2}\right)\geq\frac{r}{2}\left(1-\frac{1}{2}-\frac{1}{3}\right)>0.$$

This completes the proof.

Combining theorem 2 and lemma 3 yields:

COROLLARY 4: *If Γ is an H-group of genus 0 then there is an even integer $r\geq 4$ such that $C^0(\Gamma,-r,1)$ contains a cusp form with no zero in $\mathcal{H}$. Denote this cusp form by $\Delta(\Gamma;z)$.*

EXAMPLES: 1. For $\Gamma=\Gamma(2)$, $e=0$ and $t=3$. Thus by lemma 3 we may choose $r=6$. In this case we may choose

$$\Delta(\Gamma;z)=\eta^{12}(z).$$

2. For $\Gamma=\Gamma_\vartheta$, $e=1$, $l_1=2$, and $t=2$. Thus $r=8$. In fact we have

$$\Delta(\Gamma;z)=\eta^{8/3}(2z)\eta^{8/3}(z/2)\eta^{32/3}\left(\frac{z+1}{2}\right).$$

3. For $\Gamma=\Gamma^2$, the subgroup of $\Gamma(1)$ generated by the squares of elements in $\Gamma(1)$, $e=2$, with $l_1=l_2=3$, and $t=1$. Thus $r=6$ and

$$\Delta(\Gamma;z)=\eta^{12}(z).$$

4. For $\Gamma=\Gamma^3$, the subgroup of $\Gamma(1)$ generated by the cubes of elements of $\Gamma(1)$, $t=1$ and $e=3$, with $l_1=l_2=l_3=2$. Thus $r=4$ and in fact

$$\Delta(\Gamma;z)=\eta^{8}(z).$$

4. A Special Hecke Basis Theorem

In section 5 we present a proof of the Hecke basis theorem for the groups $\Gamma(\lambda_n)$ (theorem 1). Here we state and prove a very special, but simple result of the same type.

THEOREM 5: *Suppose r is a positive even integer. Then a Hecke basis for $C^0(\Gamma,-r,1)$ exists in the following four cases:*

(i) $g=0$, $t=1$, $e=3$, $l_1=l_2=l_3=2$
(ii) $g=0$, $t=1$, $e=2$, $l_1=2, l_2=3$
(iii) $g=0$, $t=1$, $e=2$, $l_1=2, l_2=4$
(iv) $g=0$, $t=1$, $e=2$, $l_1=l_2=3$.

REMARK: Groups with these signatures are respectively (i) Γ^3, (ii) $\Gamma(1)$, (iii) $\Gamma(\sqrt{2})$ and (iv) Γ^2.

PROOF: A simple check shows that in the four cases listed the function $\Delta(\Gamma;z)$ of corollary 4 has exactly one zero in a fundamental domain of Γ. This is a simple zero at the parabolic cusp $i\infty$. We indicate the proof in case (iii) only, the proofs in the other cases being similar. In this case $\Delta(\Gamma;z)$ has dimension -8. Thus we consider the functions

$$\Delta^l(\Gamma;z)\cdot E_{r-8l}(\Gamma,1;z),\qquad r-8l\geq 0,\qquad r-8l\neq 2,\ l=1,2,\ldots. \tag{4.1}$$

Since $\dim C^0(\Gamma,-2,1)=0$, we may restrict our attention to $r\geq 4$. Then the number of functions given by (4.1) is $[r/8]$ if $r\not\equiv 2 \pmod 8$ and $[r/8]-1$ if $r\equiv 2 \pmod 8$. Thus it suffices to prove that $\dim C^0(\Gamma,-r,1)$ is also $[r/8]$ or $[r/8]-1$ according as $r\not\equiv 2 \pmod 8$ or $r\equiv 2 \pmod 8$. But this last fact is a simple consequence of (3.4) with the $\kappa_h=0$ and $g=0$.

REMARK: It is not too hard to prove that the four cases listed in theorem 5 are the only cases in which the function $\Delta(\Gamma; z)$ of corollary 4 has exactly one zero.

5. Proof of Theorem 1

For the group $\Gamma(\lambda_n)$, $n \geq 3$, we have $g=0$, $t=1$, $e=2$, and $l_1=2$, $l_2=n$. We shall write $\Delta(z)$ for $\Delta(\Gamma(\lambda_n); z)$ in the remainder of the proof. If n is even $\Delta(z)$ has degree $-2n$ and a zero of order $(n-2)/2$ at $i\infty$; if n is odd $\Delta(z)$ has degree $-4n$ and a zero of order $n-2$ at $i\infty$. Since $\Delta(z)$ has no zero in $\mathscr{H}$ we may form

$$\Delta_1(z) = \begin{cases} \Delta(z)^{2/(n-2)} & \text{if } n \text{ is even} \\ \Delta(z)^{1/(n-2)} & \text{if } n \text{ is odd.} \end{cases}$$

Then, in every case, $\Delta_1(z) \in C^0(\Gamma(\lambda_n), -4n/(n-2), v_1)$ where v_1 is a multiplier system corresponding to the group $\Gamma(\lambda_n)$ and the degree $-4n/(n-2)$, and, in addition $\Delta_1(z)$ has a zero of order one at $i\infty$. Since $\Delta_1(i) \neq 0$, we conclude that $v_1(S)=1$, $v_1(T)=e^{-2\pi i n/(n-2)}$, where

$$S=\begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix}, \qquad \lambda_n = 2\cos\frac{\pi}{n}, \qquad \text{and} \qquad T=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We want to investigate Hecke bases for the spaces $C^0(\Gamma(\lambda_n), -k, 1)$, where k is an even integer ≥ 4. (Recall that $\dim C^0(\Gamma(\lambda_n), -2, 1)=0$.)

We introduce another cusp form

$$\Delta_2(z) = \Delta_1(z)^{k(n-2)/4n},$$

which is of degree $-k$ on $\Gamma(\lambda_n)$ and, like $\Delta(z)$ and $\Delta_1(z)$, has no zeros in $\mathscr{H}$. It follows that the multiplier system v_2 connected with $\Delta_2(z)$ satisfies

$$v_2(S) = e^{2\pi i k(n-2)/4n}, \qquad v_2(T) = e^{-\pi i k/2}. \tag{5.1}$$

Thus $\Delta_2(z)$ has multiplier system identically 1 if and only if $4|k$ and $4n|k(n-2)$. If these conditions hold, then $\Delta_2(z) \in C^0(\Gamma(\lambda_n), -k, 1)$.

We consider the functions

$$E_{k-\frac{4nl}{n-2}}(\Gamma(\lambda_n), v_1^{-l}; z) \cdot \Delta_1^l(z), \qquad 0 < l < \frac{(k-2)(n-2)}{4n}. \tag{5.2}$$

The inequality is imposed upon the integer l to insure that the subscript $k-\dfrac{4nl}{n-2}$ is greater than 2, and thus that the Eisenstein series $E_{k-\frac{4nl}{n-2}}$ exists in the usual sense of absolute convergence. Since $v_1(S)=1$ it is also true that $v_1^{-l}(S)=1$; consequently the function $E_{k-\frac{4nl}{n-2}} \cdot \Delta_1^l$ of (5.2) has a zero at the cusp $i\infty$ of order precisely l. Thus the functions in (5.2) are linearly independent. If the conditions $4|k$ and $4n|k(n-2)$ hold we add $\Delta_2(z)$ to the set of functions (5.2). The question to be investigated is whether the resulting set of functions is a basis for $C^0(\Gamma(\lambda_n), -k, 1)$. Note that $\Delta_2(z)$ has a zero at $i\infty$ of order $k \cdot \dfrac{n-2}{4n} > \dfrac{(k-2)(n-2)}{4n}$. Thus when $\Delta_2(z)$ is included it is linearly independent of the functions (5.2).

Of obvious importance is the case when $4n \mid (k-2)(n-2)$, that is, when $l=(k-2)(n-2)/4n$ is an integer. Then $k-4nl/(n-2)=2$ and the Eisenstein series in question is $E_2(\Gamma(\lambda_n), v_1^{-l}; z)$. By work of Petersson this Eisenstein series exists (in a sense weaker than that of absolute convergence) provided that v_1^l is not identically 1. Since $v_1^l(S)=1$, we consider

$$v_1^l(T) = \exp\left(\frac{-2\pi i n}{n-2} \cdot \frac{(k-2)(n-2)}{4n}\right) = \exp\left(\frac{-\pi i(k-2)}{2}\right).$$

Thus $v_1^l(T)=1$ if and only if $4\,|\,(k-2)$. If $4n\,|\,(k-2)(n-2)$, but $4\nmid(k-2)$, then, we have available the function $E_2(\Gamma(\lambda_n), v_1^{-l}; z)\cdot\Delta_1^l(z)$, $l=(k-2)(n-2)/4n$, an element of $C^0(\Gamma(\lambda_n), -k, 1)$ with zero at $i\infty$ of order exactly $(k-2)(n-2)/4n$.

Put $\rho(k, n)=[(k-2)(n-2)/4n]$. Since $4n|k(n-2)$ and $4n|(k-2)(n-2)$ cannot hold simultaneously there are three mutually exclusive and jointly exhaustive cases:

(i) If $4\,|\,(k-2)$ and $4n\,|\,(k-2)(n-2)$ we have $\rho(k, n)-1$ linearly independent functions.

(ii) If $4\,|\,k$ and $4n\,|\,k(n-2)$ we have $\rho(k, n)+1$ linearly independent functions.

(iii) Otherwise we have $\rho(k, n)$ linearly independent functions.

We now turn to the calculation of $\mu(k, n)=\dim C^\infty(\Gamma(\lambda_n), -k, 1)$. The formula (3.4) implies that

$$\mu(k, n)=k\left(\frac{q}{2}-1\right)-a_1/2-a_2/n,\quad q=3-\frac{1}{2}-\frac{1}{n}.$$

Since the multiplier system here is identically 1, $a_1=0$ or 1 according as $4\,|\,k$ or $4\nmid k$, and a_2 is chosen to satisfy $a_2\equiv-\frac{k}{2}\pmod n$, $0\leq a_2<n$. We have, then,

$$\mu(k, n)=k\left(\frac{1}{4}-\frac{1}{2n}\right)-\frac{a_1}{2}-\frac{a_2}{n},\tag{5.3}$$

with $a_1\equiv-\frac{k}{2}\pmod 2$, $0\leq a_1<2$ and $a_2\equiv-\frac{k}{2}\pmod n$, $0\leq a_2<n$. The following is an immediate consequence of (5.3).

PROPOSITION 6: $\mu(k, n)=\left[\frac{k}{4}\right]+\left[\frac{-k}{2n}\right]$.

In order to determine when (that is, for which combinations of k and n) we have constructed a complete Hecke basis for $\Gamma(\lambda_n)$, we now compare $\mu(k, n)$, and $\rho(k, n)$. Toward this end write $k=4ns+2p$, with s an integer ≥ 0 and $0\leq p<2n$. Then it is immediate that

$$\mu(k, n)=(n-2)s+\left[\frac{p}{2}\right]-\left[\frac{-p}{n}\right]$$

and

$$\rho(k, n)=(n-2)s+\left[\frac{p-1}{2}-\frac{p-1}{n}\right],$$

and we need only compare the numbers $\left[\frac{p}{2}\right]+\left[\frac{-p}{n}\right]$ and $\left[\frac{p-1}{2}-\frac{p-1}{n}\right]$, with $0\leq p<2n$. A straightforward calculation involving the consideration of various ranges of p yields a result summarized in table 1.

The conditions $4\,|\,(k-2)$, $4n\,|\,(k-2)(n-2)$ are equivalent to $4\,|\,(k-2)$, $2n\,|\,(k-2)$, which are in turn equivalent to the conditions p odd, $n\,|\,(p-1)$. Since $0\leq p<2n$, the latter two conditions hold if and only if $p=1$ or $p=n+1$ with n even. Thus we have constructed $\rho(k, n)-1$ linearly independent elements of $C^\infty(\Gamma(\lambda_n), -k, 1)$ precisely when $p=1$ or $p=n+1$, with n even. A glance at our table reveals that we have exactly the correct number of functions whenever p is odd, and thus for odd p we have constructed a Hecke basis.

TABLE 1

Restriction on p	Value of $\mu(k, n)-\rho(k, n)$
$p=0$	1
p even, $0<p\leq n/2+1$	0
p even, $n/2+1<p\leq n$	1
p even, $n<p\leq\frac{3}{2}n+1$	0
p even, $\frac{3}{2}n+1<p<2n$	1
$p=1$	−1
p odd, $p=n+1$ (i.e. n even)	−1
p odd, $p\neq 1$, $p\neq n+1$	0

The conditions $4|k$, $4n|k(n-2)$ are equivalent to $4|k$, $2n|k$, which are in turn equivalent to p even, $n|p$. These hold if and only if $p=0$ or $p=n$ with n even. Thus we have $\rho(k, n)+1$ linearly independent functions if and only if $p=0$ or $p=n$ with n even. Our table shows us that when p is even we have achieved a Hecke basis except when $n/2+1<p<n$ or $\frac{3}{2}n+1<p<2n$. In these latter two ranges of p we are exactly one function short of a basis. This completes the proof of theorem 1.

6. The Missing Function

It is easy to see that the "missing function" in the exceptional cases of theorem 1 is one with highest possible order zero at $i\infty$. Although in these cases we cannot express this function in the form (4.2) we can easily construct it from any given basis $\varphi_1, \ldots, \varphi_\mu$ of $C^0(\Gamma(\lambda_n), -k, 1)$. (Here we are assuming that $\mu=\dim C^0(\Gamma(\lambda_n), -k, 1)>0$.) In fact we give a simple and uniform construction for a cusp form in $C^0(\Gamma(\lambda_n), -k, 1)$ with maximal order zero at $i\infty$ that is valid in *all* cases, including the nonexceptional cases (those in which theorem 1 gives us a complete Hecke basis). Afterwards we show that in the nonexceptional cases the function we here construct agrees with the element of highest order zero, already constructed in the Hecke basis, up to a constant multiple.

By (3.4) we have

$$\mu=\mu(k, n)=k\left(\frac{1}{4}-\frac{1}{2n}\right)-\frac{a_1}{2}-\frac{a_2}{n}.$$

On the other hand, we conclude from (2.10) that the total number of zeros of an element of $C^0(\Gamma(\lambda_n), -k, 1)$ in a fundamental domain of $\Gamma(\lambda_n)$ is

$$N=k\left(\frac{1}{4}-\frac{1}{2n}\right).$$

At i, the fixed point of T, a cusp form in $C^0(\Gamma(\lambda_n), -k, 1)$ has a zero of order congruent to $\frac{a_1}{2}$ modulo 1; at ρ, the fixed point of ST, such a cusp form has a zero of order congruent to $\frac{a_2}{n}$ modulo 1. Thus the maximum order zero at $i\infty$ of an element of $C^0(\Gamma(\lambda_n), -k, 1)$ is

$$N-\frac{a_1}{2}-\frac{a_2}{n}=\mu(k, n)=\mu.$$

We now proceed as in the proof of theorem 2 to form a linear combination

$$\sum_{j=1}^{\mu}\zeta_j\varphi_j(z)=\sum_{m=1}^{\infty}\left(\sum_{j=1}^{\mu}\zeta_j b_m(j)\right)e^{2\pi imz/\lambda_n}$$

of the basis elements $\varphi_1, \ldots, \varphi_\mu$. The homogeneous system

$$\sum_{j=1}^{\mu}\zeta_j b_m(j),\ 1\leq m\leq \mu-1$$

has a nontrivial solution, so that the element of $C^0(\Gamma(\lambda_n), -k, 1)$ so constructed has a zero of order exactly μ *at* $i\infty$.

Suppose now we have $f_1, f_2\in C^0(\Gamma(\lambda_n), -k, 1)$, both with zeros of maximal order μ at $i\infty$. Then both f_1 and f_2 have zeros of order $a_1/2$ and a_2/n at i and ρ, respectively. It follows that f_1/f_2 is an automorphic *function* on $\Gamma(\lambda_n)$ with no poles and no zeros in a fundamental domain for $\Gamma(\lambda_n)$, and thus f_1/f_2 is a nonzero constant [3]. This shows the essential uniqueness of an element of $C^0(\Gamma(\lambda_n), -k, 1)$ with maximal order zero at $i\infty$.

7. Parametrization Theorems

In this section we parametrize entire automorphic forms and multiplier systems in some special but important cases. The derivations are based on the relations (2.10), $N=r\left(g-1+\frac{q}{2}\right)$, for the total number of zeros in a fundamental region $\mathscr{R}$, of an entire form, and (3.5) $\sum_{h=1}^{t}\kappa_h+\sum_{m=1}^{e}a_m/l_m\equiv N \bmod 1$. (2.10) can be obtained by integrating F'/F around the boundary of $\mathscr{R}$; (3.5) is a consequence of (2.10) and definitions of the order of $F(z)$ at the points of $\mathscr{R}$. The usual way to obtain $\mu=\dim C^+(\Gamma, -r, v)$ is to apply Petersson's Riemann-Roch Theorem (3.4). However, this is much deeper than the above mentioned results. Some of the cases that can be handled by this method are given in the table that follows. We feel that this type of derivation may prove useful in the classroom when there is insufficient time to derive (3.4). The parametrization in the case of $\Gamma(1)$ is known [8]; however we feel that it is a very good illustration of the method. The parametrization in the case of Γ_ϑ appears to be new. Other cases which can be handled in this way are summarized in table 2.

We consider the case of $\Gamma=\Gamma(1)$, the modular group, which has the presentation

$$E_1=-T=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\qquad E_2=-TU^{-1}=\begin{pmatrix}0&1\\-1&1\end{pmatrix},\qquad P=U=\begin{pmatrix}1&1\\0&1\end{pmatrix} \tag{7.1}$$

with

$$E_1^2=E_2^3=E_1E_2P=-I\,.$$

If r is a real number and v a multiplier system for $\Gamma(1)$ and $-r$, let $\kappa=\kappa_1$, a_1, and a_2 be as in section 2. We can prove the following theorem which we might call a Hecke basis theorem for entire forms.

THEOREM 7: *If* $F\epsilon C^+(\Gamma(1), -r, v)$ *then there are complex constants* c_j *such that*

$$F(z)=\eta^{24\kappa}(z)\sum c_j\Delta^j(z)E_{2k-12j}(z) \tag{7.2}$$

and the summation j *is over the integers* 0, 1, 2, . . ., [2k/12] *with* $2k-12j\neq 2$, *where* $2k=r-12\kappa$ *is an even integer.*

If we let v_r denote the multiplier system of $\eta^{2r}(z)$, then we can state the following:

COROLLARY 8: *With* F *expressible in the form* (7.2) *we have* $v=v_{12\kappa}$. *From this it follows that*

$$v(M)=e[24\kappa\Phi(M)],\qquad M\epsilon\Gamma(1), \tag{7.3}$$

where

$$\Phi(M)=\begin{cases}b/24d-sgn\,(d-1)/4 & \text{if}\quad c=0\\ \frac{1}{2}[(a+d)/12c-(sgn\,c)s(d,|c|)-sgn\,c/4] & \text{if}\quad c\neq 0,\end{cases} \tag{7.4}$$

for $M=\begin{pmatrix}a&b\\c&d\end{pmatrix}$, *and*

$$s(h,k)=\sum_{\mu=1}^{k-1}\left(\frac{\mu}{k}-\frac{1}{2}\right)\left(\frac{\mu h}{k}-\left[\frac{\mu h}{k}\right]-\frac{1}{2}\right) \tag{7.5}$$

is Dedekind's sum [10].

This result is given in Lehner [4].

COROLLARY 9:

$$\dim C^+(\Gamma(1), -r, v)=\begin{cases}\left[\frac{r}{12}-\kappa\right]+1 & \text{if}\quad r-12\kappa\not\equiv 2(12)\\ \left[\frac{r}{12}-\kappa\right] & \text{if}\quad r-12\kappa\equiv 2(12).\end{cases} \tag{7.6}$$

Let $\{\Gamma, -r, v\}$ denote the meromorphic automorphic forms with multiplier system v, and of degree $-r$.

COROLLARY 10: *If* $F \in \{\Gamma(1), -r, v\}$, *then*

$$F(z) = R(j(z))G(z), \tag{7.7}$$

where $R(x)$ *is a rational function,* $G \in C^+(\Gamma(1), -r, v)$, *and* $j(z)$ *is the modular invariant* $E_{12}(z)/\Delta(z)$.

PROOF OF THEOREM 7: Let κ, a_1, a_2 be as before, with v a multiplier system on $\Gamma(1)$ for the degree $-r$. Then $\kappa + a_1/2 + a_2/3 \equiv r/12 \mod 1$. Let $G(z) = F(z)/\eta^{24\kappa}(z)$. Then $G \in C^+(\Gamma(1), -(r-12\kappa), v')$. Let κ', a_1', a_2' be the values for v' corresponding to κ, a_1 and a_2 for v. Then $\kappa' = 0$ and $a_1'/2 + a_2'/3 \equiv (r-12\kappa)/12 \pmod 1$. Thus $r - 12\kappa$ is an even integer $2k$. It also follows from this relation that $v'(E_1) = v'(E_2) = 1$. This is proved by writing $v'(E_1) = e^{\pi i k + 2\pi i a_1'/2}$, $v'(E_2) = e^{\pi i 2k/3 + 2\pi i a_2'/3}$. Then $a_1'/2 + a_2'/3 \equiv 2k/12 \mod 1$ is uniquely solvable mod 12 for a_1', a_2' and, in fact, $2a_m' + 2k \equiv 0 \mod 2(m+1)$, $m = 1, 2$. For example $k = 1$, $a_1' = 1$, $a_2' = 2$. Thus $v' \equiv 1$, and therefore $G \in C^+(\Gamma(1), -2k, 1)$. Now using Hecke's basis theorem, we find

$$G(z) = \sum \Delta^j(z) E_{2k-12j}(z),$$

where summation is over $j = 0, 1, \ldots, [2k/12]$ and $2k - 12j \neq 2$. The corollaries are immediate consequences of the theorem, the transformation formula of $\log \eta(z)$, and the well-known fact that a modular function is a rational function of $j(z)$.

As a second example we take Γ to be Γ_ϑ, the subgroup of the modular group generated by $T = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$ and $U^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Γ_ϑ has the presentation

$$E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad E^2 = EP_1P_2 = -I. \tag{7.8}$$

Let r be a real number and v a multiplier system for Γ_ϑ and $-r$, and suppose

$$v(P_j) = e^{2\pi i \kappa_j} \qquad \text{and} \qquad v(E) = e^{\pi i r/2 + 2\pi i a/2}, \tag{7.9}$$

where $0 \leq \kappa_j < 1$ and $a = 0$ or 1. Let

$$h_{\alpha\beta}(z) = \eta^\alpha(2z)\eta^\alpha(z/2)\eta^\beta\left(\frac{z+1}{2}\right). \tag{7.10}$$

Then, from the transformation formula of $\log \eta(z)$,

$$\begin{aligned} h_{\alpha\beta}(Ez) &= e((2\alpha+\beta)/8)(-z)^{\alpha+\beta/2}h_{\alpha\beta}(z) \\ h_{\alpha\beta}(P_1z) &= e((\alpha+2\beta)/24)(-z+2)^{\alpha+\beta/2}h_{\alpha\beta}(z) \\ h_{\alpha\beta}(P_2z) &= e((5\alpha+\beta)/24)h_{\alpha\beta}(z). \end{aligned} \tag{7.11}$$

If we let $v_{\alpha\beta}$ denote the multiplier system for $h_{\alpha\beta}(z)$, then $h_{\alpha\beta}(z) \in \{\Gamma_\vartheta, -(\alpha+\beta/2), v_{\alpha\beta}\}$. Let

$$\psi(z) = h_{16/3, -8/3}(z). \tag{7.12}$$

By (7.11), $\psi(z) \in C^+(\Gamma_\vartheta, -4, 1)$ and $\psi(i\infty) \neq 0$, $\psi(1) = 0$.

LEMMA 11: (i) *Let* $r \equiv 0 \mod 4$, $r > 0$; *then*

$$\psi^j(z)E_{r-4j}(z, I), \qquad j = 0, 1, \ldots, [r/4],$$

is a basis for $C^+(\Gamma_\vartheta, -r, 1)$.

(ii) *Let* $r \equiv 2 \bmod 4$, $r > 2$; *then*

$$E_r(z, A),\ \psi^j(z)E_{r-4j}(z, I),\ j=0, 1, \ldots, \left[\frac{r}{4}\right]-1,$$

is a basis for $C^+(\Gamma_\vartheta - r, 1)$.

NOTE: The Eisenstein series $E_{2k}(z, I)$ and $E_{2k}(z, A)$ are defined to be the sums

$$\frac{1}{2}\sum_{\substack{(c,d)\equiv(0,1)\\ \bmod 2}} (cz+d)^{-2k} \quad \text{and} \quad \frac{1}{2}\sum_{\substack{(c,d)\equiv(1,0)\\ \bmod 2}} (cz+d)^{-2k},$$

respectively. We use the facts that $E_{2k}(i\infty, I)=1$, $E_{2k}(1, I)=0$, $E_{2k}(i\infty, A)=0$, $E_{2k}(1, A)=1$ [9].

PROOF: Let $F \in C^+(\Gamma_\vartheta, -r, 1)$. Then F has $r/4$ zeros in a fundamental region for Γ_ϑ. In the case $r \equiv 0 \bmod 4$, let $c_0, c_1, \ldots, c_s$, $s=r/4$, be determined from the expansion of F in the local variable $t=e^{2\pi iz/2}$ so that

$$F(z)=\sum_{j=0}^{s} c_j\psi^j(z)E_{r-4j}(z)$$

has a zero of order $s+1$ at $i\infty$. But $s+1 > r/4$, so the difference is identically zero.

In the second case determine b so that $G(z)=F(z)-bE_r(z, A)$ has a zero in the local variable t at $z=1$. Then determine $c_0, c_1, \ldots, c_{s-1}$, where $r=4s+2$, as above, so that $H(z)=G(z) - \sum_{j=0}^{s-1} c_j\psi^j(z)E_{r-4j}(z, I)$ has a zero at $i\infty$ of order s in the local variable $t=e^{2\pi iz/2}$. From $H(-1/z) = z^rH(z)$, we see that $H(i)=0$, so that $H(z)$ has order at least $1/2$ at $z=i$. In total $H(z)$ has at least $s+1+\frac{1}{2} > \frac{r}{4}$ zeros. Thus the difference is identically zero.

COROLLARY 12: *dim* $C^+(\Gamma_\vartheta, -r, 1) = [r/4]+1$ *when* r *is an even integer larger than* 2.

THEOREM 13: *Let* $F(z) \in C^+(\Gamma_\vartheta, -r, v)$. *Then* $F(z)/h_{\alpha\beta}(z) \in C^+(\Gamma_\vartheta, -(r-(\alpha+\beta/2)), 1)$, *where* $\alpha+2\beta=24\kappa_1$, $5\alpha+\beta=24\kappa_2$, *and* $r-\alpha-\frac{\beta}{2}$ *is an even integer.*

REMARKS: One can consider theorem 13 a Hecke basis-type theorem in view of lemma 11. For example, if $r-\alpha-\beta/2$ is a multiple of 4, then the elements in $C^+(\Gamma_\vartheta, -r, v)$ are linear combinations of $h_{\alpha\beta}(z)\psi^j(z)E_{r-4j}(z, I)$.

PROOF: Let $F \in C^+(\Gamma_\vartheta, -r, v)$ and α and β as in the statement of the theorem. Then $G=F/h_{\alpha\beta} \in C^+(\Gamma_\vartheta, -(r-\alpha-\beta/2, v'))$. If we let κ_1', κ_2', and a' be as in (7.10), with v replaced by v' and r replaced by $r'=r-\alpha-\beta/2$, then $\kappa_1'=\kappa_2'=0$ and $a'/2 \equiv r'/4 \pmod 1$, so r' is even. We see with with these replacements in (7.10) that $v' \equiv 1$.

We conclude this section with a table containing parametrizations of automorphic forms and multiplier systems on various groups. The entries $E_r(z)$ denote Eisenstein series of degree $-r$ for the group in question.

TABLE 2

Group	Presentation	$\Delta(\Gamma, z)$	Multiplier system v for Γ and $-r$	Basis for $C^+(\Gamma, -r, v)$	Dimension $C^+(\Gamma, -r, v)$
$\Gamma(1)$ modular group	$E_1 = -T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $E_2 = -TU^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ $P = U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $E_1^2 = E_2^3 = E_1E_2P = -I$	$\Delta(z) = \eta^{24}(z)$	$v(P) = e(\kappa)$ with $0 \leq \kappa < 1$ $v = v_{24\kappa}$ where v_a is the M.S. of $\eta^a(z)$ $v(V) = e(24\kappa\Phi(V))$ where $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $\Phi(V) = \begin{cases} b/24d - \operatorname{sgn}(d-1)/4 \text{ if } c = 0 \\ \frac{1}{2}\{(a+d)/12c - (\operatorname{sgn} c)\cdot s(d,\lvert c\rvert) \\ \qquad - (\operatorname{sgn} c)/4\} \text{ if } c \neq 0 \end{cases}$ $s(h, k)$ is Dedekind's sum: $\sum_{\mu=1}^{k-1}\left(\frac{\mu}{k} - \frac{1}{2}\right)\left(\frac{\mu h}{k} - \left[\frac{\mu h}{k}\right] - \frac{1}{2}\right)$ [4][10]	$\eta^{24\kappa}(z)\Delta^j(z)E_{2k-12j}(z)$, where $r = 12\kappa + 2k$, and $j = 0, 1, \ldots, [2k/12]$, $2k - 12j \neq 2$ [8]	$\left[\frac{r}{12} - \kappa\right] + 1$ if $r - 12\kappa \not\equiv 2(12)$, and $\left[\frac{r}{12} - \kappa\right]$ if $r - 12\kappa \equiv 2(12)$.
Γ^2 $[\Gamma(1):\Gamma^2] = 2$	$E_1 = -U^{-1}T = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ $E_2 = -TU^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ $P = U^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $E_1^3 = E_2^3 = E_1E_2P = -I$	$\eta^{12}(z)$	$v(P) = e(\kappa) 0 \leq \kappa < 1$ and v_a is M.S. for $\eta^a(z)$ then $v = v_{12\kappa}\chi$ where χ is an even character on $\Gamma^2 (\chi(-I) = 1)$.	ASSUMPTION: $v = 1$ and r even integral. $\eta^{12j}(z)E_{r-6j}(z)$, where $j = 0, 1, \ldots, [r/6]$, and $r - 6j \neq 2$.	(If $v = 1$, r even) $\left[\frac{r}{6}\right] + 1$ if $r \not\equiv 2(6)$, and $\left[\frac{r}{6}\right]$ if $r \equiv 2(6)$.
Γ^3 $[\Gamma(1):\Gamma^3] = 3$ $\Gamma^3 \lhd \Gamma(1)$	$E_1 = -U^{-1}TU = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ $E_2 = -T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $E_3 = UTU^{-1} = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}$ $P = U^3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ $E_1^2 = E_2^2 = E_3^2 = -I$ $E_1E_2E_3P = I$	$\eta^8(z)$	$v(P) = e(\kappa)$ $0 \leq \kappa < 1$ and v_a is M.S. for $\eta^a(z)$ then $v = v_{8\kappa}\chi$ where χ is an even character on Γ^3.	ASSUMPTION: $v = 1$ and r even integral. $\eta^{8j}(z)E_{r-4j}(z)$, where $r \equiv 0(4)$, and $j = 0, 1, \ldots, [r/4]$, and $r - 4j \neq 2$.	(If $v = 1$, r even) $\left[\frac{r}{4}\right] + 1$ if $r \equiv 0(4)$ $\left[\frac{r}{4}\right]$ if $r \equiv 2(4)$.
Γ_δ $[\Gamma(1):\Gamma_\delta] = 3$	$E = -T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $P_1 = -TU^2 = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$ $P_2 = U^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $E_1^3 = EP_1P_2 = -I$	$\zeta\eta^3(z) \times$ $\eta^3\left(\frac{z+1}{2}\right)$ $\zeta = e(-1/6)$ [9]	$v(P_j) = e(\kappa_j)$ $0 \leq \kappa_j < 1$ $j = 1, 2$ $v = v_{\alpha\beta}$ where $v_{\alpha\beta}$ is M.S. for $h_{\alpha\beta}(z) = \eta^\alpha(2z)\eta^\alpha\left(\frac{z}{2}\right)\eta^\beta\left(\frac{z+1}{2}\right)$ and $\alpha + 2\beta = 24\kappa_1$, $5\alpha + \beta = 24\kappa_2$. $v(V) = e(\alpha\Phi(V_1) + \alpha\Phi(V_2) + \beta\Phi(V_3))$ where $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $V_3 = \begin{pmatrix} a+c & (b+d-a-c)/2 \\ 2c & d-c \end{pmatrix}$, and if $c \equiv 0(2)$	1) $h_{\alpha\beta}(z)E_4^j(z, A)E_{r'-4j}(z, I)$, $j = 0, \ldots, \left[\frac{r'}{4}\right] - 1$, if $r' = r - \alpha - \beta/2 \equiv 0(4)$. 2) $h_{\alpha\beta}(z)E_{r'}(z, A)$, $h_{\alpha\beta}(z)E_4^j(z, A)E_{r'-4j}(z, I)$, $j = 0, \ldots, \left[\frac{r'}{4}\right] - 1$, if $r' \equiv 2(4)$.	$\left[\frac{r}{4} - \kappa_1 - \kappa_2\right] + 1$

TABLE 2—Continued

Group	Presentation	$\Delta(\Gamma, z)$	Multiplier system v for Γ and $-r$	Basis for $C^+(\Gamma, -r, v)$	Dimension $C^+(\Gamma, -r, v)$
			$V_1 = \begin{pmatrix} a & 2b \\ c/2 & d \end{pmatrix}$, $V_2 = \begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}$. If $c \equiv 1(2)$ $V_1 = \begin{pmatrix} 2a & b \\ c & d/2 \end{pmatrix}$ and $V_2 = \begin{pmatrix} a/2 & b \\ c & 2d \end{pmatrix}$ Φ defined in entry for $\Gamma(1)$.		
$\Gamma(2)$ $[\Gamma(1):\Gamma(2)] = 6$	$P_1 = -TU^2T = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ $P_2 = (T^{-1}U^{-2})^2 = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$ $P_3 = U^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $P_1P_2P_3 = -I$	$\eta^{12}(z)$	$v(P_j) = e(\kappa_j)$ $0 \leq \kappa_j < 1$ $j = 1, 2, 3$ $v = v_{\alpha\beta\gamma}$ the M.S. for $h_{\alpha\beta\gamma}(z) = \eta^\alpha(2z)\eta^\beta\left(\frac{z}{2}\right)\eta^\gamma\left(\frac{z+1}{2}\right)$ where $4\alpha + \beta + \gamma = 24\kappa_1$, $\alpha + 4\beta + \gamma = 24\kappa_2$ and $\alpha + \beta + 4\gamma = 24\kappa_3$. $v(V) = e(\alpha\Phi(V_1) + \beta\Phi(V_2) + \gamma\Phi(V_3))$ where V, V_1, V_2, V_3 are the same as in the entry for Γ_θ with $c \equiv 0(2)$. [5], [13]	Let $\psi(z) = \eta^{20/3}(2z)\eta^{-4/3}(z/1) \times \eta^{-4/3}\left(\frac{z+1}{2}\right)$ and $\varphi(z)$ $= \eta^{-4/3}(2z)\eta^{20/3}(z/2)\eta^{-4/3}\left(\frac{z+1}{2}\right)$. Then φ, $\psi \in C^+(\Gamma(2), -2, 1)$, $\psi(i\infty) = 0$ and $\varphi(0) = 0$. $h_{\alpha\beta\gamma}(z)\psi^m(z)\varphi^n(z)$, with $m, n \geq 0$ *and* $2m + 2n = r - (\alpha + \beta + \gamma)/2 \equiv 0(2)$.	$r/2 - (\kappa_1 + \kappa_2 + \kappa_3) + 1$
$\Gamma(3)$ $[\Gamma(1):\Gamma(3)] = 24$	$-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $P_1 = U^3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ $P_2 = -TU^3T = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$ $P_3 = A^{-1}U^3A = \begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix}$ $P_4 = B^{-1}U^3B = \begin{pmatrix} -5 & 12 \\ -3 & 7 \end{pmatrix}$ $(-I)^2 = I$, $P_1P_2P_3P_4 = I$ $A = TU^{-1}$, $B = T^{-1}U^{-2}$	$\eta^8(z)$	$v(P_j) = e(\kappa_j)$, $0 \leq \kappa_j < 1$, $j = 1, 2, 3, 4$ $v = v_{\alpha\beta\gamma\delta}$ the M.S. for $h_{\alpha\beta\gamma\delta}(z) = \eta^\alpha(3z)\eta^\beta\left(\frac{z}{3}\right)\eta^\gamma\left(\frac{z+1}{2}\right)\eta^\delta\left(\frac{z+2}{3}\right)$ where $-4\alpha = -11\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4$, $-4\beta = \kappa_1 - 11\kappa_2 + \kappa_3 + \kappa_4$, $-4\gamma = \kappa_1 + \kappa_2 - 11\kappa_3 + \kappa_4$ and $-4\delta = \kappa_1 + \kappa_2 + \kappa_3 - 11\kappa_4$. [13]	Let $\psi(z) = h_{11/4, -1/4, -1/4, -1/4}(z)$ and $\varphi(z) = h_{-1/4, 11/4, -1/4, -1/4}(z) \in C^+(\Gamma(3), -1, \chi)$ where $\chi = \pm 1$, $\chi(P_i) = 1$, $\chi(-I) = -1$. Then $\psi(i\infty) = 0$, $\varphi(0) = 0$. $h_{\alpha\beta\gamma\delta}(z)\varphi^m(z)\psi^n(z)$ where $m, n \geq 0$, and $m + n = r - \kappa_1 - \kappa_2 - \kappa_3 - \kappa_4$.	$r - \kappa_1 - \kappa_2 - \kappa_3 - \kappa_4 + 1$
$\Gamma(\sqrt{2})$	$E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $E_2 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ -1 & \sqrt{2} \end{pmatrix}$ $P = \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix}$ $E_1^2 = E_2^4 = -I = E_1E_2P$ $V = \begin{pmatrix} a & b\sqrt{2} \\ c\sqrt{2} & d \end{pmatrix}$ called even $V = \begin{pmatrix} a\sqrt{2} & b \\ c & d\sqrt{2} \end{pmatrix}$ called odd	$\eta^8(\sqrt{2}z) \times \eta^8(z/\sqrt{2})$	$v(V) = v_{s,t}v(V, \sqrt{2}, -r)$ where $v(V, \sqrt{2}, -r)$ is M.S. for $[\eta(\sqrt{2}z)\eta(z/\sqrt{2})]^r$ If V is *even* $v_{s,t}(V) = e((bd - ac + 2bc)t/4)$, and $v(V, \sqrt{2}, -r) = e(-r \operatorname{sgn}\frac{c}{4} + r\Phi(V_1) + r\Phi(V_2))$ where $V_1 = \begin{pmatrix} a & 2b \\ c & d \end{pmatrix}$ and $V_2 = \begin{pmatrix} a & b \\ 2c & d \end{pmatrix}$. Φ is defined in the entry for $\Gamma(1)$. If V is *odd* $v_{s,t}(V) = e(s + t(ac - bd - 2ad)/4)$, and	$\nu(P) = e(\kappa)$, $r' = r - 8\kappa$ $\eta^{8\kappa}(z, \sqrt{2})\eta^{8j}(z, \sqrt{2})E_{r'-8j}(z, v_{0,0})$ when $(a_1, a_2) = (0, 0)$, $(1, 3)$, $(0, 2)$ or $(1, 1)$. $\eta^{8\kappa}(z, \sqrt{2})\eta^{8\kappa}(z, \sqrt{2})E_{r'-8j}(z, v_{1,0})$ when $(a_1, a_2) = (1, 2)$, $(0, 1)$, $(1, 0)$, $(0, 3)$ where $v(E_1) = e^{\pi i r/2 + 2\pi i a_1/2}$ $v(E_2) = e^{\pi i r/4 + 2\pi i a_2/4}$, and $j = 0, 1, \ldots, \left[\frac{r'}{8}\right]$, $r' - 8j \neq 2$. [14]	$\left[\frac{r}{8} - \kappa\right] + 1$ when $r' = r - 8\kappa \neq 2(4)$. $\left[\frac{r}{8} - \kappa\right]$ if $r' \equiv 2(4)$ and $(a_1, a_2) = (1, 3)$ The case $(a_1, a_2) = (0, 1)$ is excluded in the above formula.

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			$v(V, -\sqrt{2}, -r) = e(-r \operatorname{sgn} \frac{c}{4} + r\Phi(V_3) + r\Phi(V_4))$ where $V_3 = \begin{pmatrix} 2a & b \\ c & d \end{pmatrix}$ and $V_4 = \begin{pmatrix} a & b \\ c & 2d \end{pmatrix}$. [2], [14]		
$\Gamma(\sqrt{3})$	$E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $E_2 = \begin{pmatrix} 0 & 1 \\ -1 & \sqrt{3} \end{pmatrix}$ $P = \begin{pmatrix} 1 & \sqrt{3} \\ 0 & 1 \end{pmatrix}$ $E_1^3 = E_2^6 = -I = E_1E_2P$ $V = \begin{pmatrix} a & b\sqrt{3} \\ c\sqrt{3} & d \end{pmatrix}$ even $V = \begin{pmatrix} a\sqrt{3} & b \\ c & d\sqrt{3} \end{pmatrix}$ odd	$\eta^{12}(\sqrt{3}z) \times \eta^{12}(z/\sqrt{3})$	$v(V) = v_{s,t}(V)v(V, \sqrt{3}, -r)$ where $v(V, \sqrt{3}, -r)$ is M.S. for $[\eta(\sqrt{3}z)\eta(z/\sqrt{3})]^r$. If V is *even* $v_{s,t}(V) = e(t(bd - ac + 3bc)/6)$, and $v(V) = e(-r \operatorname{sgn} \frac{c}{4} + r\Phi(V_1) + r\Phi(V_2))$. If V is *odd* $v_{s,t}(V) = e\left(\frac{s}{2} + t(ac - bd + 3ad)/6\right)$, and $v(V, \sqrt{3}, -r) = e(-r \operatorname{sgn} \frac{c}{4} + r\Phi(V_3) + r\Phi(V_4))$ where V_1, V_2, V_3, V_4 are as in entry for $\Gamma(\sqrt{2})$ with 3 replacing 2. [2], [14]	$v(P) = e(\kappa)$, $r' = r - 6\kappa$ 1) $r' \equiv 0(2)$, $\eta^{6\kappa}(z, \sqrt{3})\eta^{6j}(z, \sqrt{3})E_{r'-6j}(z, v_{s-j,0})$ where $s \equiv r'/2 + a_1 \bmod 2$ and $v(E_1) = e^{2\pi i r'/4 + 2\pi i a_1/3}$, 2) $r' \equiv 1(2)$, $\eta^{6\kappa}(z, \sqrt{3})\eta^{6j}(z, \sqrt{3})E_{r'-6j}(z, v_1 v_{s-j,s})$ where $s \equiv (r' - 1)/2 + a_1(2)$, a_1 as above and v_1 is the M.S. for $\eta(z, \sqrt{3})$ [14]	$\left[\frac{r}{6} - \kappa\right] + 1$ if $r' = r - 6\kappa \not\equiv 2(6)$, and $\left[\frac{r}{6} - \kappa\right]$ if $r' \equiv 2(6)$.

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The Distribution of Eigenvalues of Covariance Matrices of Residuals in Analysis of Variance

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A rigorous definition is given for the concept of an "interaction matrix" (Z_{ij}) where $i = 1$ to m and $j = 1$ to n, in terms of two idempotent matrices A_r and B_s of rank r and s, respectively. It is then shown that the frequency distribution of the eigenvalues of $(Z)(Z)'$ depends only on r and s. Applications are given to matrices of residuals arising from two-way data, either by removing row and/or column-means, or by applying any number of sweeps of the "vacuum cleaner." The theorems are important in the theory of the analysis of two-way tables of nonadditive data.

Key words: Analysis of variance; covariance matrix; eigenvalues; interaction; matrix; residuals; two-way table; vacuum cleaner.

1. Introduction

In a recent paper [1],[1] a method has been presented for a partitioning of the row by column interaction in two-way tables and the practical usefulness of this method as a tool in data analysis has been demonstrated. The method required the calculation, by Monte Carlo techniques, of quantities that are analogous to the "degrees of freedom" in ordinary analysis of variance. These new degrees of freedom were calculated as follows.

Let (Z_{ij}) be an $m \times n$ matrix of independent random normal deviates (zero mean, unit variance). Consider the $m \times n$ matrix of residuals (d_{ij}), where

$$d_{ij} = Z_{ij} + Z_{..} - Z_{i.} - Z_{.j},$$

where a dot indicates averaging over the index replaced by the dot.

Then the new quantities, denoted as degrees of freedom, are the expected values of the eigenvalues of the matrix product $(d)(d)'$ (where $'$ indicates the transpose of a matrix). The expected values of these eigenvalues are functions of m and n, and were tabulated as such. Professor John Tukey, of Princeton University, has pointed out that the degrees of freedom calculated by this procedure are of far more general applicability than indicated in the paper in question. The present paper provides the theoretical development from which the more general results are obtained. It is based on a new, mathematically rigorous definition for an interaction matrix, from which an important general result can be derived.

2. A Definition of Interaction of Two-Way Tables

Let (d_{ij}) be an $m \times n$ matrix of normally distributed variates, all of zero mean; and let the covariance of d_{ij} and $d_{i'j'}$ be denoted by $\mathrm{cov}(d_{ij}, d_{i'j'})$,

We will say that (d_{ij}) in an *interaction matrix of* r *degrees of freedom by* s *degrees of freedom* if $\mathrm{cov}(d_{ij}, d_{i'j'})$ is of the form

$$\mathrm{cov}(d_{ij}, d_{i'j'}) = [a_{ii'} \cdot b_{jj'}]\, \sigma^2,$$

[1] Figures in brackets indicate the literature references at the end of this paper.

where the $m \times m$ matrix $A_r \equiv (a_{ii'})$ is idempotent of rank r, and the $n \times n$ matrix $B_s \equiv (b_{jj'})$ is idempotent of rank s.

We will say that A_r and B_s are *the covariance matrices associated with* the interaction matrix (d_{ij}).

3. The Eigenvalues of (d)(d)′

THEOREM 1: *If* (d_{ij}) *is an* $m \times n$ *interaction matrix with associated covariance matrices* A_r *and* B_s, *then the nonzero eigenvalues of the matrix product* (d)(d)′ (*where* (d)′ *denotes the transpose of* (d)) *are the same as those of the matrix* (t)(t)′, *where* (t_{ij}) *is an* $r \times s$ *matrix of normally and independently distributed variates* t_{ij}, *of zero mean and common variance* σ^2.

PROOF: Let $P = (p_{ii'})$ represent the $m \times m$ orthogonal matrix whose rows are the eigenvectors of A_r. Since A_r is idempotent and of rank r, we have

$$PA_rP' = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right), \tag{2}$$

where I_r is the identity matrix or rank r. Similarly, representing by $Q = (q_{jj'})$ the $n \times n$ orthogonal matrix whose rows are the eigenvectors of B_s, we have

$$QB_sQ' = \left(\begin{array}{c|c} I_s & 0 \\ \hline 0 & 0 \end{array}\right). \tag{3}$$

From (2) and (3) we derive, respectively,

$$\sum_k \sum_l p_{ik} p i' a_{kl} = \begin{cases} \delta_{ii'} & \text{for } i, i' \leq r \\ 0 & \text{for } i \text{ or } i' > r \end{cases} \tag{4}$$

and

$$\sum_u \sum_v q_{ju} q_{j'v} b_{uv} = \begin{cases} \delta_{jj'} & \text{for } j, j' \leq s \\ 0 & \text{for } j \text{ or } j' > s, \end{cases} \tag{5}$$

where δ_{kl} is the Kronecker delta ($\delta_{kl} = 1$ for $k = l$, and $\delta_{kl} = 0$ for $k \neq l$).

Consider now the transformation

$$(t^*_{ij}) = P(d_{ij})Q'. \tag{6}$$

Then,

$$(t^*)(t^*)' = P(d)Q'Q(d)'P'$$

or

$$(t^*)(t^*)' = P(d)(d)'P'. \tag{7}$$

From (7) it follows that $(t^*)(t^*)'$ and $(d)(d)'$ have the same nonzero eigenvalues. Furthermore, we have, on account of (6)

$$t^*_{ij} = \sum_k^m \sum_u^n p_{ik} q_{ju} d_{ku}. \tag{8}$$

Consequently,

$$\operatorname{cov}(t^*_{ij}, t^*_{i'j'}) = \sum_k^m \sum_u^n \sum_l^m \sum_v^n p_{ik} q_{ju} p_{i'l} q_{j'v} \operatorname{cov}(d_{ku}, d_{lv}). \tag{9}$$

Introducing (1) into (9), we obtain

$$\begin{aligned} \operatorname{cov}(t^*_{ij}, t^*_{i'j'}) &= \sigma^2 \sum_k^m \sum_u^n \sum_l^m \sum_v^n p_{ik} q_{ju} p_{i'l} q_{j'v} a_{kl} b_{uv} \\ &= \sigma^2 \left(\sum_k \sum_l p_{ik} p_{i'l} a_{kl} \right) \left(\sum_u \sum_v q_{ju} q_{j'v} b_{uv} \right) \end{aligned}$$

which, as a result of eqs (4) and (5), becomes:

$$\operatorname{cov}\,(t^*_{ij}, t^*_{i'j'}) = \begin{cases} \sigma^2 \cdot \delta_{ii'} \cdot \delta_{jj'} & \text{for} \quad \begin{cases} i, i' \leq r \\ j, j' \leq s \end{cases} \\ 0 & \text{for} \quad \begin{cases} i \text{ or } i' > r \\ j \text{ or } j' > s. \end{cases} \end{cases} \tag{10}$$

From (8) it also follows that the t^*_{ij} are normally distributed with $E(t^*_{ij}) = 0$ for all i and j. Consequently, because of (10), the t^*_{ij} are equal to zero for all $i, i' > r$ or $j, j' > s$. For $i, i' \leq r$ and $j, j' \leq s$, it follows from (10) that the t^*_{ij} are normally and independently distributed with zero mean and variance σ^2.

We have already shown that $(d)(d)'$ and $(t^*)(t^*)'$ have the same nonzero eigenvalues. If we omit from the matrix (t^*), all elements for which $i > r$, or $j > s$ (all of which are zero), we obtain an $r \times s$ matrix, say (t_{ij}), such that $(t)(t)'$ has the same nonzero eigenvalues as $(t^*)(t^*)'$, and consequently as $(d)(d)'$. *QED.*

NOTE: We have assumed, in the definition of an interaction matrix, that the d_{ij} are *normally* distributed, and have proved that *under this assumption*, the t^*_{ij} defined by eq (8) (except for those that are identically zero), are *independently* and *normally* distributed with zero mean and variance σ^2. We can generalize the definition of an interaction matrix by requiring only that the d_{ij} be *conditionally* normal, *given* the $a_{ii'}$ and $b_{jj'}$. Since the distribution of the t_{ij} does not depend on the $a_{ii'}$ and $b_{jj'}$, it is *unconditionally* normal with zero mean and variance σ^2, and Theorem I remains true under the more general definition of an interaction matrix.

4. Residuals from an Additive Structure

THEOREM II: *Let* $(\mathrm{d_{ij}})$ *be a matrix whose elements are*

$$\mathrm{d_{ij}} = y_{ij} - \hat{\mu} - \hat{\rho}_i - \hat{\gamma}_j \qquad \begin{cases} \mathrm{i} = 1 \text{ to m} \\ \mathrm{j} = 1 \text{ to n} \end{cases} \tag{11}$$

where $\mathrm{y_{ij}}$ *are independent normal variates, such that* $\mathrm{E(y_{ij})} = \mu + \rho_i + \gamma_j$, *and whose common variance is* σ^2, *and where* $\hat{\mu}$, $\hat{\rho}_i$, $\hat{\gamma}_j$ *are the usual estimates of the grand mean, the row effects and the column effects. Then* $(\mathrm{d_{ij}})$ *is an interaction matrix of* $(\mathrm{m}-1)$ *degrees of freedom by* $(\mathrm{n}-1)$ *degrees of freedom.*

PROOF: It is readily shown from (11) that

$$\operatorname{cov}\,(d_{ij}, d_{i'j'}) = \left(\delta_{ii'} - \frac{1}{m}\right)\left(\delta_{jj'} - \frac{1}{n}\right)\sigma^2. \tag{12}$$

The matrix $A = \left(\delta_{ii'} - \frac{1}{m}\right)$ is idempotent. Its trace is $\sum_i \left(\delta_{ii'} - \frac{1}{m}\right) = m - 1$. Hence its rank is $m - 1$. Similarly, the matrix $B = \left(\delta_{jj'} - \frac{1}{n}\right)$ is idempotent, of rank $n - 1$. This proves the theorem.

5. Residuals from the Vacuum Cleaner [2]

LEMMA I: *Let* $(\mathrm{y_{ij}})$ *be an* $\mathrm{m} \times \mathrm{n}$ *matrix of observations* $\mathrm{y_{ij}}$, *with common variance* σ^2. *Let* $(\mathrm{d_{ij,k}})$ *be the matrix of residuals obtained from the* $\mathrm{y_{ij}}$ *after extraction of the grand mean* $(\hat{\mu})$, *the row effects* $(\hat{\rho}_i)$, *the column effects* $(\hat{\gamma}_j)$, *and* k *sweeps of the vacuum cleaner* [3]. *Then the following relation holds:*

$$\mathrm{d_{ij,k}} = \mathrm{y_{ij}} - (\hat{\mu} + \hat{\rho}_i + \hat{\gamma}_j) - \sum_{l=1}^{k} [K_l p_{il} q_{lj} + \alpha_{il} q_{lj} + \beta_{lj} p_{il}] \tag{13}$$

[2] The results contained in Lemmas I, II, and III are not novel. They have been derived previously, by a different method, by W. L. Nicholson [2], and are presented here only for completeness.

in which

$$(1)\quad \left.\begin{aligned} \sum_i p_{ii} &= \sum_j q_{ij} = 0 \\ \sum_i p_{ii}^2 &= \sum_j q_{ij}^2 = 1 \end{aligned}\right\} \qquad \text{for all } l.$$

$$(2)\quad p_{ii} = \frac{\rho_i}{\sqrt{\sum_i \rho_i^2}} \qquad q_{ij} = \frac{\gamma_j}{\sqrt{\sum_j \gamma_j^2}}$$

$$(3)\quad p_{ii} = \frac{\alpha_{i-1,i}}{\sqrt{\sum_i \alpha_{i-1,i}^2}}, \qquad q_{ij} = \frac{\beta_{i-1,j}}{\sqrt{\sum_j \beta_{i-1,j}^2}}$$

$$(4)\quad \begin{cases} K_i = \sum_i \sum_j d_{ij,i-1} p_{ii} q_{ij} \\ \alpha_{ij} = \sum_j d_{ij,i-1} q_{ij} - K_i p_{ii} \\ \beta_{ij} = \sum_i d_{ij,i-1} p_{ii} - K_i q_{ij}. \end{cases}$$

These relations result from the definition of the vacuum-cleaner process (see [3]), according to which

$$d_{ij,\,l} = d_{ij,\,l-1} - [K_l p_{il} q_{jl} + \alpha_{il} q_{jl} + \beta_{jl} p_{il}], \tag{14}$$

and the quantities K, p, q, α, β, are calculated as above.

LEMMA II: *Using the notation of Lemma I, we have*

$$\left.\begin{aligned} \sum_i p_{il} p_{il'} &= \delta_{ll'} \\ \sum_j q_{jl} q_{jl'} &= \delta_{ll'} \end{aligned}\right\} \quad \textit{for all } l, l' \qquad \begin{matrix} (15) \\ \textit{and} \\ (16) \end{matrix}$$

PROOF: Let $(t) = (t_1, t_{21} \ldots, t_n)$ be any vector of n elements, such that $\sum_j t_j = 0$, and assume that the following relations hold for some given value of l.

$$\left.\begin{aligned} (1)\ \sum_j t_j q_{l+1,j} &= 0 \\ (2)\ \sum_j t_j d_{ij,l} &= 0 \end{aligned}\right\} \qquad \text{for all } i,$$

where $d_{ij,l}$ satisfies eq. (14) of Lemma I.

Then the following relations hold:

$$(a)\ \sum_j t_j q_{l+2,\,j} = 0$$

$$(b)\ \sum_j t_j d_{ij,\,l+1} = 0 \qquad \text{for all } i.$$

PROOF:

$$(a)\ \sum_j t_j q_{l+2,j} = \sum_j t_j \frac{\beta_{l+1,j}}{\sqrt{\sum \beta^2_{l+1,j}}} = \frac{1}{\sqrt{\sum_j \beta^2_{l+1,j}}} \sum_j t_j \left[\sum_i d_{ij,l} p_{l+1,i} - K_{l+1} q_{l+1,j} \right]$$

$$= \frac{1}{\sqrt{\sum_j \beta^2_{l+1,j}}} \left[\sum_i p_{l+1,i} \left(\sum_j t_j d_{ij,l} \right) - K_{l+1} \sum_j t_j q_{l+1,j} \right] = 0, \text{ because of (1) and (2).}$$

$$(b)\ \sum_j t_j d_{ij,l+1} = \sum_j t_j d_{ij,l} - K_{l+1,}\, p_{l+1,i} \sum_j t_j q_{l+1,j}$$

$$- \alpha_{l+1,i} \sum_j t_j q_{l+1,j} - p_{l+1,i} \sum_j t_j \beta_{l+1,j} = 0, \text{ because of (1), (2), and (a).}$$

Thus, if conditions (1) and (2) are fulfilled for any value of l, they are also fulfilled for any value $l' > l$. In that case, we have

$$\sum t_j q_{rj} = 0 \qquad \text{for } l' \geqslant l+1.$$

If we now make $t_j \equiv q_{l,j}$, we can verify that conditions (1) and (2) are fulfilled. Hence,

$$\sum_j q_{lj} q_{rj} = 0 \qquad \text{for } l' \neq l,$$

and by definition $\sum_j q_{lj} q_{lj} = 1$. The relation $\sum_i p_{il} p_{ri} = \delta_{lr}$ is proved in a similar way.

LEMMA III: *Defining* $(d_{ij,k})$ *as in Lemma I, we have*

$$cov\ (d_{ij,k}, d_{i'j',k}) = \sigma^2 \left(\delta_{ii'} - \frac{1}{m} - \sum_{l=1}^{k} p_{il} p_{il'} \right) \left(\delta_{jj'} - \frac{1}{n} - \sum_{l=1}^{k} q_{lj} q_{lj'} \right).$$

PROOF: The lemma is easily verified for $k=1$. It is also readily shown, using Lemma II, that if the lemma holds for k, it also holds for $k+1$. Thus, the lemma is proved by mathematical induction on k.

THEOREM III: *Define* $(d_{ij,k})$ *as in Lemma I, and assume furthermore that the* y_{ij} *are normally and independently distributed, with* $E(y_{ij}) = \mu + \rho_i + \gamma_j$, *and common variance* σ^2; *then the* $m \times n$ *matrix* $(d_{ij,k})$ *is an interaction matrix of* $(m-k-1)$ by $(n-k-1)$ *degree of freedom.*

PROOF: The theorem results at once from Lemma III amd the fact that a matrix of the form

$$\left(\delta_{ii'} - \frac{1}{m} - \sum_{l=1}^{k} p_{il} p_{il'} \right),$$

with

$$\sum_i p_{il} p_{ri} = \delta_{lr},$$

is idempotent and of rank $m-k-1$. (The trace of this matrix is

$$\sum_i \left[\delta_{ii} - \frac{1}{m} - \sum_{l=1}^{k} p_{il}^2 \right] = m - m\left(\frac{1}{m}\right) - k = m-k-1.)^3$$

[3] The distribution of the $d_{ij,k}$ is only *conditionally* normal, given all p_{ik} and q_{lj} values, but as shown on p. 151, this does not invalidate Theorem I. Consequently, Theorem III is true.

THEOREM IV: *Let* (y_{ij}) *be an* $m \times n$ *matrix of normal, independent variates* y_{ij}, *whose means are* $E(y_{ij}) = \mu = \rho_i + \gamma_j$ *and whose common variance is* σ^2. *Extract from* y_{ij} *the mean* $\hat{\mu}$, *the row effect* $\hat{\rho}_i$, *and the column effect* $\hat{\gamma}_j$, *as well as* k *sweeps of the vacuum cleaner, and denote by* $(d_{ij,k})$ *the matrix of residuals resulting from this treatment.*

Then the nonzero eigenvalues of the matrix product $(d_{ij,k})(d_{ij,k})'$ are the same as those of $(t)(t)'$ where (t) is an $(m-k-1)$ by $(n-k-1)$ matrix (t_{ij}) of *normal, independent* variates t_{ij} of zero mean and common variance σ^2.

The theorem holds for all values of k, from $k=0$ to $k = \min[(m-1), (n-1)]$.

PROOF: The theorem is an immediate consequence of Theorems I and III.

6. Principal Result and Summary

As indicated in the introduction, the present work was motivated by a desire to generalize results obtained in connection with a new method of data analysis. We are now in a position to state this generalization in the following precise form.

THEOREM V: Let (d_{ij}) be an $m \times n$ interaction matrix of r degrees of freedom by s degrees of freedom. Then the probability distribution of the eigenvalues of the matrix product (d)(d)′ depends only on r and s, and not on m and n.

PROOF: The theorem follows at once from Theorem I.

Our principal result is the following: To study the probability distribution of the eigenvalues of $(d)(d)'$, where (d) is *any* interaction matrix with r by s degrees of freedom, it is sufficient to study the probability distribution of the eigenvalues of $(t)(t)'$, where (t) is an r by s matrix whose elements are independent random normal deviates (mean zero, unit variance). The probability distribution in question is completely determined by the values of r and s.

The author is greatly indebted to Professor John Tukey for pointing out to him the generality of his previous results and thereby providing the incentive for the present work.

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Expansions With Coefficient Algorithms for Time Domain Responses of Skin Effect Lossy Coaxial Cables

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Time domain step response expansions with coefficient algorithms are developed from the system function of a doubly terminated skin effect lossy coaxial transmission line. Three models of series impedance valid for (a) high frequency, (b) high and intermediate frequency, (c) high and low frequency, are incorporated into the system function. The system function is expanded via the method of polynomial expansions of analytic functions through generating relations and inverted term by term into the time domain through the inverse Laplace transform. Step responses for time domain reflectometry and transmission are developed, computed, and compared with experimental results.

Key words: Coefficient algorithms; expansions; inverse Laplace transform; reflectometry; skin effect; step response; system function; transmission.

1. Introduction

In the analysis of skin effect lossy transmission lines one encounters propagation, characteristic impedance, and reflection functions which depend on $\sqrt{s}$. For example, $\exp\{-x\gamma(s)\}$, $Z_0(s)$, and $\rho(s)=(Z-Z_0(s))/(Z+Z_0(s))$ depend on terms like $K\sqrt{s}$ or $R+K\sqrt{s}$. Such functions appear in the study of pulse transmission in coaxial cables and integrated circuit transmission line structures, etc., see references [1, 2, 3, 4, 5, 6].[1]

A method of solution in the class of problems for $R\neq 0$ is to affect rearrangements of the power series representing $\gamma(s)$, $Z_0(s)$, and $\rho(s)$. A series representation for $\exp(-x\gamma(s))$ is realized through the technique of polynomial expansions via generating relations. Term by term Laplace transform inversion is then possible and the resulting functional series with certain expansion coefficient algorithms is the time domain solution.

The time domain expansions which were obtained in this paper were specifically developed for the study of the pulse response of coaxial cables [6]. The conclusions of that study demonstrated that the planar skin effect transmission line series impedance models, the computational results, and the experimental data correlated quite well.

We are now ready to pose the principal problem: determine the pulse distortion of a doubly terminated coaxial cable with skin effect losses. A circuit model of the system appears in figure 1 for a transmission line of length l with

(a) $Z_{s0}(s)$, the source impedance;
(b) $Z_r(s)$, the receiving end impedance;
(c) $E_1(s)$, the voltage step generator;
(d) $E_2(s, 0)$ and $E_2(s, l)$, step responses at input and output transmission line terminals respectively.

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[1] Figures in brackets indicate the literature references at the end of this paper.

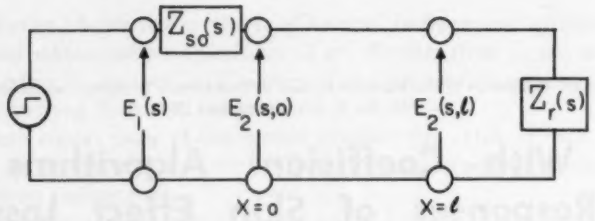

FIGURE 1. *Circuit model of system function.*

Proceeding to the system function [6]

$$\frac{E_2(s,x)}{E_1(s)}=\frac{Z_0(s)}{Z_0(s)+Z_{s0}(s)}\,\frac{\epsilon^{-x\gamma(s)}+\rho_r(s)\epsilon^{-(2l-x)\gamma(s)}}{1-\rho_{s0}(s)\rho_r(s)\epsilon^{-2l\gamma(s)}} \tag{1.1}$$

observe the voltage reflection function definitions

$$\rho_{s0}(s)=\frac{Z_{s0}(s)-Z_0(s)}{Z_{s0}(s)+Z_0(s)} \quad \text{and} \quad \rho_r(s)=\frac{Z_r(s)-Z_0(s)}{Z_r(s)+Z_0(s)} \tag{1.2}$$

Now $\gamma(s)$ and $Z_0(s)$, propagation and characteristic impedance functions respectively, define as follows [6]:

$$\gamma(s)=\sqrt{Z(s)Y(s)}, \tag{1.3}$$

$$Z_0(s)=\sqrt{\frac{Z(s)}{Y(s)}}. \tag{1.4}$$

The functions $Z(s)$ and $Y(s)$ which are unit length transmission line parameters are known as series impedance and shunt admittance respectively. They define via the relations [6]

$$Y(s)=Cs \tag{1.5}$$

and

$$Z(s)=Ls+Ks^{1/2}+R \tag{1.6}$$

where K denotes the planar skin effect parameter.

The purpose of this paper is to obtain time domain step response expansions and expansion coefficient alogorithms from the system function (1.1) for three pairs of terminating impedances: (a) $Z_{s0}(s)=0$ and $Z_r(s)=Z_0(s)$, (b) $Z_{s0}(s)=\sqrt{L/C}$ and $Z_r(s)=+\infty$, (c) $Z_{s0}(s)=Z_r(s)=\sqrt{L/C}$. We are interested in step responses at $x=0$ and/or $x=l$ for small and large time arguments.

In conjunction with the last remark observe that form (1.6) contains three models for series impedance [6] on selecting the proper value of R: (a) $R=0$, high frequency; (b) $R=R'/4$, high and intermediate frequency; (c) $R=R'$, high and low frequency. R' is the direct current (dc) resistance of the inner and outer conductors per unit length.

2. Properties and Expansions of $\gamma(s)$ and $Z_0(s)$

The following properties are important for extracting time domain expansions in later sections.

(a) *Singularity property.* Let L_1 be the defined s domain of the inverse Laplace transform i.e., $-\pi<\arg s\leq\pi$. In view of (1.3), (1.4), (1.5) select the positive branch of $s^{1/2}$. Now keeping in mind that $L>0$, $K\geq 0$, $R\geq 0$ observe $Ls+Ks^{1/2}+R=0$ possesses roots $s_0^{1/2}$ and $s_1^{1/2}$ in the left hand plane. Therefore, $\arg s_k\notin L_1 (k=0 \text{ or } 1)$ and $\gamma(s)$ (or $Z_0(s)$) has one branch point $s=0$.

(b) *Bound on* $|\epsilon^{-l\gamma(s)}|$. We want to show that $|\epsilon^{-l\gamma(s)}|<1$ for all interior points in the region comprised of the right-half plane (rhp). According to Titchmarsh [7] the maximum-modulus theorem is valid for either branch of $\exp\{-l\gamma(s)\}$ provided $|\exp\{-l\gamma(s)\}|$ is single valued at the origin. To see this is true let

$$\gamma(\delta+j\omega)=|\gamma(\delta+j\omega)|\,\epsilon^{-j\varphi(\delta,\omega)}.$$

Then clearly $\exp(-|\gamma(\delta+j\omega)|\cos\varphi(\delta,\omega))\to\exp(0)=1$ independently of the path. Now on the rhp boundary $|e^{-l\gamma(s)}|$ has a maximum equal to 1 and a mimimum equal to 0. Therefore, $|e^{-l\gamma(s)}|<1$ for $\mathrm{Re}\ s>0$.

(c) *Lower bound on* $ReZ_0(s)$. Let $Z_0(s)=|Z_0(s)|\,\epsilon^{j\varphi_0(s)}$ and observe from (1.3), (1.4) and (b) above that $-\pi/2\leq\varphi(\delta,\omega)\leq\pi/2$ implies $-\pi/2\leq\varphi_0(\delta,\omega)\leq\pi/2$. Hence, $\mathrm{Re}\ Z_0(\delta+j\omega)=|Z_0(\delta+j\omega)|\cos\varphi_0(\delta,\omega)\geq 0$.

(d) *Expansions.* It is convenient to set $sl\sqrt{LC}=\sigma$ in obtaining a binomial expansion of $\gamma(s)$. Then

$$l\gamma(s)\equiv\hat{\gamma}(\sigma)=\sigma\left\{1+\sum_{n=0}^{\infty}(-1)^n c_n\left\{\frac{\beta_0+\beta_2\sigma^{1/2}}{\sigma}\right\}^{n+1}\right\}, \tag{2.1}$$

$$\left.\begin{aligned} c_n&=\left|\frac{1}{2}\left(-\frac{1}{2}\right)\cdots\left(\frac{1}{2}-n\right)\right|\frac{1}{(n+1)!},\\ \beta_0&=\frac{R}{L}l\sqrt{LC},\qquad \beta_2=\frac{K}{L}l^{1/2}(LC)^{1/4}. \end{aligned}\right\} \tag{2.2}$$

Implementing the binomial theorem in (2.1) produces

$$\begin{aligned}\hat{\gamma}(\sigma)=\sigma\Bigg\{1&+c_0\sigma^{-1}\underset{0}{\{}d^{(1)}_{1,0}+d^{(1)}_{1,1}\sigma^{1/2}\underset{0}{\}}-c_1\sigma^{-2}\underset{0}{\{}d^{(1)}_{2,0}+d^{(1)}_{2,1}\sigma^{1/2}\\&+d^{(1)}_{2,2}\sigma^{2/2}\underset{0}{\}}+\cdots+(-1)^{n+m-1}c_{n+m-1}\sigma^{-2\frac{(n+m)}{2}}\sum_{k=0}^{n+m}d^{(1)}_{n+m,k}\sigma^{k/2}\\&+(-1)^{n+m}c_{n+m}\sigma^{-\frac{2(n+m+1)}{2}}\sum_{k=0}^{n+m+1}d^{(1)}_{n+m+1,k}\sigma^{k/2}+\cdots\Bigg\}\underset{1}{}\end{aligned} \tag{2.3}$$

where

$$d^{(1)}_{n,k}=\frac{n!}{(n-k)!k!}\beta_0^{n-k}\beta_2^{k}.$$

This series converges absolutely when $\infty<|s|<S_2=\left\{\frac{K}{2L}+\sqrt{\frac{K^2}{4L^2}+\frac{R}{L}}\right\}^2$ i.e., a proof of the bound appears in appendix A. Hence, rearrangement of the series (2.3) is realized by recognizing that $n+1$ terms contribute $\sigma^{\frac{-2n}{2}}\left(\sigma^{\frac{-2n+1}{2}}\right)$. We require that integral value of k in (2.3) which cancels σ^{-m} $(\sigma^{-m-1/2})$ and obtain the rearranged series

$$\hat{\gamma}(\sigma)=\sigma\left\{1+\sum_{n=0}^{\infty}a^{\{1\}}_{1,n+1}\sigma^{-\frac{n+1}{2}}\right\}, \tag{2.4}$$

with coefficient algorithms

$$a^{\{1\}}_{1,2n}=(-1)^{n-1}\sum_{m=0}^{n}(-1)^m c_{n+m-1}d^{(1)}_{n+m,2m}, \tag{2.5a}$$

$$a^{\{1\}}_{1,2n+1}=(-1)^{n}\sum_{m=0}^{n}(-1)^m c_{n+m}d^{(1)}_{n+m+1,2m+1}. \tag{2.5b}$$

Similarly an expansion of the positive branch of $\gamma(s)$ for $0 < |s| \leq S_0 <$ minimum $\left(\frac{1}{2}\frac{R}{L}, \frac{1R^2}{4K^2}\right)$ i.e., valid for small $|s|$, yields the series

$$\gamma(s) = (RC)^{1/2}s^{1/2}\left\{1+\sum_{n=0}^{\infty}(-1)^n c_n\left\{\frac{K}{R}s^{1/2}+\frac{L}{R}s\right\}^{n+1}\right\}. \tag{2.6}$$

Proceeding as in (2.3) we find the rearrangement

$$\gamma(s) = \sqrt{RC}s^{1/2}\left\{1+\sum_{n=0}^{\infty} a^{(2)}_{1,n+1}s^{\frac{n+1}{2}}\right\}, \tag{2.7}$$

with coefficients algorithms[2]

$$a^{(2)}_{1,2n} = (-1)^{n-1}\sum_{m=0}^{n}(-1)^m c_{n+m-1}d^{(2)}_{n+m,n-m}, \tag{2.8a}$$

$$a^{(2)}_{1,2n+1} = (-1)^{n}\sum_{m=0}^{n}(-1)^m c_{n+m}d^{(2)}_{n+m+1,n-m}, \tag{2.8b}$$

$$d^{(2)}_{n,k} = \frac{n!}{(n-k)!k!}\eta_0^{n-k}\eta_2^k, \qquad \eta_0 = \frac{K}{R}, \qquad \eta_2 = \frac{L}{R} \tag{2.8c}$$

3. General Method for Time Domain Solutions of the System Function

The following two inverse Laplace transforms are well known [8]:

$$\mathscr{L}^{-1}\left\{\frac{\epsilon^{-k_1\sigma^{1/2}}}{\sigma^{\frac{n}{2}+1}}\right\} = (4\tau)^{n/2}I^n \operatorname{erfc}\left(\frac{k_1}{2}\tau^{-1/2}\right) \qquad (n=0, 1, \ldots), \tag{3.1}$$

$$\mathscr{L}^{-1}\left\{s^{\frac{n-1}{2}}\epsilon^{-k_2 s^{1/2}}\right\} = \frac{\exp\left\{-\frac{k_2^2}{4t}\right\}}{2^n\sqrt{\pi}t^{n+1}}H_n\left(\frac{k_2}{2t^{1/2}}\right) \qquad (n=0, 1, \ldots). \tag{3.2}$$

Note k_1 and k_2 are positive; $I^n \operatorname{erfc} u (u = k/2t^{1/2})$ is the nth repeated integral of erfc x and possesses the recursion relation [9]

$$I^n \operatorname{erfc} u = \int_u^{\infty} I^{n-1} \operatorname{erfc} x \, dx$$

where $I^0 \operatorname{erfc} x = \operatorname{erfc} x$. The notation $H_n(u)$ denotes Hermite polynomials of degree n.

With (1.1), (3.1), (3.2) in mind we require a method of expanding $A(w) \exp(\alpha g(w))$ such that $\alpha = \alpha(x)$, and $g(w)$ yields a Taylor expansion in $\sigma^{-1/2}$ (or $s^{1/2}$). From Boas and Buck [10] the technique of polynomial expansions of analytic functions through generating relations suffices in view of the following definition:

A set of polynomials has a Sheffer representation if it is generated by the formal relation

$$A(w)\exp(\alpha g(w)) = \sum_{n=0}^{\infty} P_n(\alpha)w^n \tag{3.3}$$

[2] Superscript values of 1 and $k \geq 2$ correspond to high and low frequency expansions respectively on all coefficients.

where

$$\left.\begin{aligned} A(w) &= \sum_{n=0}^{\infty} A_n w^n, \qquad A_0 \neq 0, \\ g(w) &= \sum_{n=0}^{\infty} g_n w^n, \qquad g_1 \neq 0 \end{aligned}\right\} \tag{3.4}$$

If $g_1 = 0$ and $g_n \neq 0 (n = 2, 3, \ldots)$ we define $P_n(\alpha)$ to be a modified Sheffer polynomial. This definition clearly indicates Sheffer polynomials are rather general since they are functions of the expansion coefficients in $A(w)$, $g(w)$, exp (u). The mechanics of obtaining $P_n(\alpha)$ are taken up in section 4.

The application of (3.3) to (1.1) requires a hierarchy of coefficient algorithms such that recursive properties hold within and between algorithms. To clarify this point note the d coefficients of (2.5) require recursivity within (2.5) to determine the "a" coefficients while application of (3.3) to exp $(-\hat{\gamma}(\sigma))$ yields at least recursivity between a new algorithm and (2.5).

4. Ideal Step Response

A very simple pair of impedance values for (1.1), $Z_r(s) = Z_0(s)$ and $Z_{s0}(s) = 0$, leads to the σ domain ideal step response

$$\hat{E}_2(\sigma, l) = \hat{E} \frac{\epsilon^{-\hat{\gamma}(\sigma)}}{\sigma}, \qquad \hat{E} = El\sqrt{LC}. \tag{4.1}$$

By virtue of (3.3) this expression becomes

$$\hat{E}_2(\sigma, l) = \hat{E}\epsilon^{-a_{1,2}^{(1)}} \frac{\exp\{-(\sigma + a_{1,2}^{(1)}\sigma^{1/2}\}}{\sigma} \hat{T}_1(\sigma, l), \tag{4.2}$$

$$\hat{T}_1(\sigma, l) = \exp\left\{+\sum_{n=0}^{\infty} g_n^{(1)} \sigma^{-\left(\frac{n+1}{2}\right)}\right\}, \tag{4.3}$$

where $g_{n-3}^{(1)} = -a_{1,n}^{(1)}$ $(n \geq 3)$ and (4.3) is a member of (3.3). According to Knopp [11] the main rearrangement theorem applies to (4.3) since

$$\sum_{n=0}^{\infty} |g_n^{(1)} \sigma^{-\left(\frac{n+1}{2}\right)}| \leq M < \infty$$

i.e., (2.4) is absolutely convergent. Hence, proceeding to a tractable expansion of (4.3) obtains the rearrangement

$$\left\{\sum_{n=0}^{\infty} g_n^{(1)} \sigma^{-\left(\frac{n+1}{2}\right)}\right\}^k = \sigma^{-k/2} \sum_{n=0}^{\infty} G_{k,n,1}^{(1)} \sigma^{-n/2} \tag{4.4}$$

where all G coefficients [3] define [12] through the recursion

$$G_{k,n,1}^{(1)} = \frac{1}{ng_0^{(1)}} \sum_{p=1}^{n} \{p(k+1) - n\} g_p^{(1)} G_{k,n-p,1}^{(1)} \tag{4.5a}$$

[3] The third subscript of G means its integer value is a multiplier of each g coefficient on the right-hand side of the definition equation (4.5a). It corresponds to the number of reflections (or transmissions) which the input step voltage experiences.

with initial values [4]

$$G^{(1)}_{0,0,1} = 1, \qquad G^{(1)}_{k,0,1} = (g^{(1)}_0)^k, \qquad G^{(1)}_{1,k,1} = g^{(1)}_k. \tag{4.5b}$$

From (4.3), (4.4), and (4.5) we obtain a double power series which convolutes with respect to subscripts and yields

$$\hat{T}_1(\sigma, l) = \sum_{n=0}^{\infty} b^{(1)}_{n,1}\sigma^{-n/2}, \tag{4.6}$$

$$b^{(1)}_{n,1} = \sum_{k=1}^{n} \frac{G^{(1)}_{k,n-k,1}}{k!}, \qquad b^{(1)}_{0,1} = 1,^{5}$$

such that all coefficients $b^{(1)}_{n,1}$ are Sheffer polynomials with argument $x = l$.

Now following Doetsch [13] term by term inversion of (4.2) is permissible. Hence, in view of (3.1), (4.2), and (4.6) the time domain expression becomes

$$\hat{e}_2(\tau, l) = E\epsilon^{-a^{(1)}_{1,2}} \sum_{n=0}^{\infty} b^{(1)}_{n,1}\{4(\tau-1)\}^{n/2} I^n \operatorname{erfc}\left\{\frac{a^{(1)}_{1,1}}{2}(\tau-1)^{-1/2}\right\}. \tag{4.8}$$

To determine the τ interval of uniform convergence observe that $\sum_{n=0}^{\infty} |b^{(1)}_{n,1}| S_4^{-n/2} (S_4 > S_2 l\sqrt{LC})$ is a convergent series. If we require term by term dominance of the expansion in (4.8) the inequality

$$\{4(\tau-1)\}^{n/2} I^n \operatorname{erfc}\left\{\frac{a^{(1)}_{1,1}}{2}(\tau-1)^{-1/2}\right\} \leqslant S_4^{n/2} \tag{4.9}$$

determines the interval $1 \leqslant \tau \leqslant \tau_0$ where τ_0 is that value of τ which makes (4.9) an equality.

To compute the step response for large time arguments revert to (2.7) and consider the s domain expansion with coefficients $g^{(2)}_m = -l\sqrt{RC}a^{(2)}_{1,m+1} (m = 0, 1 \ldots)$. Then

$$E_2(s, l) = E\frac{\epsilon^{-l\gamma(s)}}{s} = E\frac{\epsilon^{-l\sqrt{RC}s^{1/2}}}{s} \exp\left\{\sum_{m=0}^{\infty} g^{(2)}_m s^{\frac{m+2}{2}}\right\}. \tag{4.10}$$

For $0 < |s| \leqslant S_0$ we use the argument supporting (4.6) to obtain

$$E_2(s, l) = E\frac{\epsilon}{s}^{-l\sqrt{RC}s^{1/2}} \sum_{n=0}^{\infty} b^{(2)}_{n,1} s^{n/2}, \tag{4.11}$$

$$b^{(2)}_{n,1} = \sum_{k=1}^{[n/2]} \frac{G^{(2)}_{k,n-k,1}}{k!}, \qquad b^{(2)}_{1,1} = 0, \tag{4.12}$$

where $[n/2]$ denotes the integer function. The coefficients $b^{(2)}_{n,1}$ are modified Sheffer polynomials with argument $x = l$.

Now inversion of (4.11) through use of (4.3) and (4.4) carries $|s|$ beyond S_0. Hence, according to Doetsch [14], we can expect the time domain expansion to be asymptotic. Indeed, this turns out to be true under rearrangement of (4.11) into odd and even numbered terms from a proof in appendix B. Hence, the conclusion contains the form

[4] Initial values of G coefficients assume the same form as (4.5b) when superscripts are changed.

[5] We define $b^{(2)}_{0,n} = 1$, independent of n and k.

$$e_2(t, l) \sim E \operatorname{erfc}\left(\frac{l\sqrt{RC}}{2t^{1/2}}\right)+\frac{E}{\pi^{1/2}} \exp\left\{\frac{-l^2RC}{4t}\right\} \times\left\langle \sum_{n=1}^{\infty} b_{n,1}^{(2)} \frac{H_{n-1}\left(\frac{l\sqrt{RC}}{2t^{1/2}}\right)}{2^{n-1}t^{n/2}}\right\rangle \tag{4.13}$$

where the operator $\langle\ \rangle$ denotes rearrangement with respect to n.

5. Open Circuit Step Response

To handle the open circuit case revert to (1.1), set $Z_{s0}(s)=\sqrt{L/C}$ and $Z_r(s)=+\infty$, and observe the input terminal step response

$$E_2(s, 0)=E_1(s) \frac{Z_0(s)}{Z_0(s)+\sqrt{L/C}} \cdot \frac{1+\epsilon^{-2l\gamma(s)}}{-\rho_{s0}(s)\epsilon^{-2l\gamma(s)}}. \tag{5.1}$$

At this point it is worthwhile to investigate the possibility of expressing the τ time domain solution of (5.1) as a sequence of reflections. This corresponds to expressing $(1-\rho_{s0}(s)\epsilon^{-2l\gamma(s)})^{-1}$ as a geometric series. The condition $|\rho_{s0}(s)\epsilon^{-2l\gamma(s)}|<1$ is satisfied in view of 3b and 3c. We remark that it is also quite easy to calculate an upper bound for $|\rho_{s0}(s)|$ by Stannard's method [15].

Applying the geometric expansion to (5.1) yields

$$\hat{E}_2(\sigma, 0)=\hat{E}_1(\sigma) \frac{\hat{Z}_0(\sigma)}{\hat{Z}_0(\sigma)+\sqrt{L/C}} \sum_{n=0}^{\infty} \hat{\rho}_{s0}^{n}(\sigma)\{\epsilon^{-2n\hat{\gamma}(\sigma)}+\epsilon^{-2(n+1)\hat{\gamma}(\sigma)}\}. \tag{5.2}$$

Following the expansion technique for $\hat{\gamma}(\sigma)$ obtains the "unreflected term" representation

$$\hat{E}_{2,0}(\sigma, 0) \equiv \frac{\hat{Z}_0(\sigma)}{\hat{Z}_0(\sigma)+\sqrt{L/C}}=\frac{1}{2}+\sum_{n=0}^{\infty} a_{2,n+1}^{(1)}\sigma^{-\left(\frac{n+1}{2}\right)}, \tag{5.3a}$$

$$a_{2,2n}^{(1)}=(-1)^{n-1} \sum_{k=0}^{n} (-1)^k c_{n+k} d_{n+k,2k}^{(1)}, \tag{5.3b}$$

$$a_{2,2n+1}^{(1)}=(-1)^{n} \sum_{k=0}^{n} (-1)^k c_{n+k+1} d_{n+k+1,2k+1}^{(1)}. \tag{5.3c}$$

Integral powers of voltage reflection functions expand into the form

$$\hat{\rho}_{s0}^{n}(\sigma)=(-2)^n \sum_{m=0}^{\infty} A_{m+1,n-1}^{(1)}\sigma^{-\left(\frac{m+n}{2}\right)} \qquad (n=1, 2, \ldots) \tag{5.4}$$

which defines itself according to the alternate recursion relations [6]

$$A_{m+1,0}^{(1)} \equiv a_{2,m+1}^{(1)} \text{ and } A_{m+1,n-1}^{(1)}=\sum_{k=0}^{m} a_{2,m+1-k}^{(1)} A_{k+1,n-2}^{(1)}. \tag{5.5}$$

We expand the exponent functions of (5.2) into the form

$$\epsilon^{-2p\hat{\gamma}(\sigma)}=\epsilon^{-2p\sigma_{1,1}^{(1)}} \exp\left\{-2p\{\sigma+a_{1,1}^{(1)}\sigma^{1/2}\}\right\} \sum_{m=0}^{\infty} b_{m,2p}^{(1)}\sigma^{-m/2} \tag{5.6}$$

[6] Of course relations (4.5) and (5.5) give the same answer. However, their implementation points out one difference which we shall mention in section 7.

such that $p=n,\ n+1$. Hence, from (5.3), (5.4), and (5.6) the nth term becomes

$$\hat{E}_{2,n}(\sigma,0)=\hat{E}(-2)^n\left\{\sum_{p=n}^{n+1}{}'\epsilon^{-2pa_{1,1}^{(1)}}\exp\frac{\{-2p\{\sigma+a_{1,1}^{(1)}\sigma^{1/2}\}\}}{\sigma}\right.$$
$$\times\left\{\frac{1}{2}\sum_{m=0}^{\infty}\left\{\sum_{k=0}^{m}b_{m-k,2p}^{(1)}A_{k+1,n-1}^{(1)}\right\}\sigma^{-\left(\frac{m+n}{2}\right)}\right.$$
$$\left.\left.+\left\{\sum_{k=0}^{m}b_{m-k,2p}^{(1)}A_{k+1,n}^{(1)}\right\}\sigma^{-\left(\frac{m+n+1}{2}\right)}\right\}\right\}\qquad(n=0,1,\ldots).\qquad(5.7)$$

The prime notation on sigma indicates the series (5.3a) replaces (5.7) at $p=0$ and, at $p=1$, $b_{m,2}^{(1)}$ replaces the sum $\sum_{k=0}^{m}b_{m-k,2}^{(1)}A_{k+1,-1}^{(1)}$.

Hence, the time domain response yields

$$e_2(\tau,0)=E\sum_{n=0}^{[\tau]}(-2)^n\left\{\sum_{p=n}^{n+1}{}'\epsilon^{-2pa_{1,1}^{(1)}}\left\{\frac{1}{2}\sum_{m=0}^{\infty}\left\{\sum_{k=0}^{m}b_{m-k,2p}^{(1)}A_{k+1,n-1}^{(1)}\right\}\right.\right.$$
$$\times\{4(\tau-2p)\}^{\frac{m+n}{2}}I^{m+n}+\sum_{m=0}^{\infty}\left\{\sum_{k=0}^{m}b_{m-k,2p}^{(1)}A_{k+1,n}^{(1)}\right\}\{4(\tau-2p)\}^{\frac{m+n+1}{2}}$$
$$\left.\left.\times I^{m+n+1}\right\}\operatorname{erfc}\{pa_{1,1}^{(1)}(\tau-2p)^{-1/2}\}\right\}\qquad(5.8)$$

and the term for $p=0$ inverts to the series

$$\hat{e}_{2,0}(\tau,0)=\frac{1}{2}+\sum_{m=0}^{\infty}a_{2,m+1}^{(1)}\frac{\tau^{(m+1)/2}}{\Gamma\left(\frac{m+3}{2}\right)}.\qquad(5.9)$$

This series converges uniformly when

$$(\tau^{m+1/2})/\Gamma\left(\frac{m+1}{2}\right)\leq S_4^{-\frac{m+1}{2}}\qquad(5.10)$$

Clearly inequality (4.9) (n replaced by $m+n$) extends to cover (5.8).

To determine the time domain asymptotic expansion let

$$\frac{Z_0(s)}{Z_0(s)+\sqrt{\frac{L}{C}}}=\left\{1+\eta_0 s^{1/2}+\eta_2 s-\eta_2^{1/2}s^{1/2}\left\{1+\sum_{n=0}^{\infty}a_{1,n+1}^{(2)}s^{(n+1)/2}\right\}\right\}\frac{1}{1+\eta_0 s^{1/2}}.\qquad(5.11)$$

Now define a set of coefficients such that

$$A_0^{(2)}=1,\qquad A_1^{(2)}=\eta_0-\eta^{1/2},\qquad A_2^{(2)}=\eta_2-\eta_2^{1/2}a_{1,1}^{(2)},\qquad A_n^{(2)}=-\eta_2^{1/2}a_{1,n-1}^{(2)}\qquad(n=3,4,\ldots).$$

Then (5.11) assumes the form

$$\frac{Z_0(s)}{Z_0(s)+\sqrt{\frac{L}{C}}}=\frac{1}{1+\eta_0 s^{1/2}}\sum_{n=0}^{\infty}A_{n,}^{(2)}\,s^{n/2} \tag{5.12}$$

In a very similar manner we obtain the expansion

$$\rho_{s0}(s)=-\frac{1}{1+\eta_0 s^{1/2}}\sum_{n=0}^{\infty}A_{n,0}^{(3)}\,s^{n/2} \tag{5.13}$$

with coefficient definitions

$$A_{0,0}^{(3)}=1,\quad A_{1,0}^{(3)}=\eta_0-2\eta_2^{1/2},\quad A_{2,0}^{(3)}=2\{\eta_2-\eta_2^{1/2}a_{1,1}^{(2)}\},\quad A_{n,0}^{(3)}=-2\eta_2^{1/2}a_{1,n-1}^{(2)}\qquad (n=3,4\ .\ .\ .).$$

For convenience redefining the coefficients of (2.7) yields

$$a_{1,1}^{(3)}=-l\sqrt{RC},\quad a_{1,n}^{(3)}=-l\sqrt{RC}\,a_{1,n-1}^{(2)}\qquad (n=2,3,\ .\ .\ .),\quad a_{1,n+1}^{(3)}=g_n^{(3)}\qquad (n=0,1,\ .\ .\ .).$$

The exponential function of (5.1) has the expansion

$$\epsilon^{-2l\gamma(s)}=\sum_{k=0}^{\infty}2^k\frac{s^{k/2}}{k!}\times\sum_{m=0}^{\infty}G_{k,m,1}^{(3)}s^{m/2}=\sum_{n=0}^{\infty}b_{n,2}^{(3)}s^{n/2}, \tag{5.14}$$

with the coefficient algorithm

$$b_{n,2}^{(3)}=\sum_{k=1}^{n}\frac{G_{k,n-k,2}^{(3)}}{k!}\qquad (n\geq 2).$$

Combining expansions (5.13) and (5.14) obtains

$$1-\rho_{s0}(s)\epsilon^{-2l\gamma(s)}=\frac{1}{1+\eta_0 s^{1/2}}\sum_{n=0}^{\infty}M_n^{(2)}s^{n/2}, \tag{5.15}$$

subject to coefficient algorithms

$$M_0^{(2)}=2,\qquad M_1^{(2)}=\eta_0+A_{1,0}^{(3)}+b_{1,2}^{(3)},$$

$$M_n^{(2)}=\sum_{k=0}^{n}A_{n-k,0}^{(3)}b_{k,2}^{(3)}\qquad (n=2,3,\ .\ .\ .).$$

Expansions (5.12) through (5.15) constitute a scheme for the series solution of (5.1) by means of the two quotients

$$E_2(s,0)=E_1(s)\left\{\frac{\sum_{n=0}^{\infty}A_n^{(2)}s^{n/2}}{\sum_{n=0}^{\infty}M_n^{(2)}s^{n/2}}+\epsilon^{-2l\sqrt{RC}s^{1/2}}\frac{\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}A_{n-k}^{(2)}b_{k,2}^{(2)}\right\}_0 s^{n/2}}{\sum_{n=0}^{\infty}M_n^{(2)}s^{n/2}}\right\}_1. \tag{5.16}$$

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Since the power series above are analytic for $0 < |s| \leq S_0$ we may perform division termwise to obtain

$$E_2(s, 0) = E_1(s) \left\{ \sum_{n=0}^{\infty} W_n^{(2)} s^{n/2} + \epsilon^{-2l\sqrt{RC}s^{1/2}} \sum_{n=0}^{\infty} W_n^{(3)} s^{n/2} \right\} \tag{5.17}$$

such that recursion formulas

$$W_n^{(2)} = \frac{A_n^{(2)} - \sum_{k=0}^{n-1} M_{n+k}^{(2)} W_k^{(2)}}{2},$$

and

$$W_n^{(3)} = \frac{\sum_{k=0}^{n-1} A_{n-k}^{(2)} b_{k,2}^{(2)} - \sum_{k=0}^{n-1} M_{n-k}^{(2)} W_k^{(3)}}{2} \qquad (n = 1, 2, \ldots),$$

with initial values $W_0^{(2)} = W_0^{(3)} = \frac{1}{2}$ enable all coefficient determinations. For a time domain expansion of (5.17) invoke the gamma function equation $\Gamma(z)\Gamma(1-z) = \pi \csc \pi z$ (z complex and nonintegral real) and observe

$$e_2(t, 0) \sim \frac{E}{2} + \frac{E}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} W_{2n-1}^{(2)} \Gamma\left(\frac{2n-1}{2}\right) t^{-\left(\frac{2n-1}{2}\right)}$$

$$+ \frac{E}{2} \operatorname{erfc}\left(\frac{l\sqrt{RC}}{2t^{1/2}}\right) + E\epsilon^{-\frac{l^2RC}{4t}} \left\langle \sum_{n=1}^{\infty} W_n^{(3)} \frac{H_{n-1}\left(\frac{l\sqrt{RC}}{2t^{1/2}}\right)}{2^{n-1}\sqrt{\pi t^n}} \right\rangle \tag{5.18}$$

as $t \to \infty$ is a complete asymptotic expansion.

The short circuit step response with impedance values $Z_r(s) = 0$ and $Z_{s0}(s) = \sqrt{\frac{L}{C}}$ is obtained from the basic σ and s domain expansions of this section.

6. Time Domain Reflectometry and Transmission

We select the impedance values $Z_{s0}(s) = Z_r(s) = \sqrt{\frac{L}{C}}$ and require a solution of (1.1) for $0 \leq x \leq l$ and $0 \leq t \leq T < \infty$. Let $\hat{\rho}_r(\sigma) = \hat{\rho}_{s0}(\sigma) \equiv \hat{\rho}(\sigma)$ and define three relations

$$\left.\begin{aligned} \alpha_{2n}(x) &= 2ln + x, \\ \alpha_{2n+1}(x) &= 2l(n+1) - x, \\ \hat{\alpha}_n(x) &= \frac{1}{l}\alpha_n(x), \ (n = 0, 1, 2 \ldots) \end{aligned}\right\} \tag{6.1}$$

which include two integer functions

$$\hat{\alpha}_n(0) = 2\left[\frac{n+1}{2}\right] \text{ and } \hat{\alpha}_n(l) = 2\left[\frac{n}{2}\right] + 1$$

at input and output terminals respectively.

Invoking the geometric expansion in (4.2) via (6.1) yields

$$\hat{E}_2(\sigma, x) = \hat{E}_1(\sigma) \frac{\hat{Z}_0(\sigma)}{\hat{Z}_0(\sigma) + \sqrt{\frac{L}{C}}} \sum_{n=0}^{\infty} \hat{\rho}^n(\sigma) \epsilon^{-\hat{\alpha}_n(x)\hat{\gamma}(\sigma)}. \tag{6.2}$$

To implement the rearranged series set

$$T_n(\sigma, x) = \exp\left\{ \hat{\alpha}_n(x) \sum_{m=0}^{\infty} g_m^{(1)} \sigma^{-\left(\frac{m+1}{2}\right)} \right\} \tag{6.3}$$

Let the definition equation

$$\left\{ \alpha_p(x) \sum_{m=0}^{\infty} g_m^{(1)} \sigma^{-\left(\frac{m+1}{2}\right)} \right\}^k = \{\hat{\alpha}_p(x)\sigma^{-1/2}\}^k \sum_{m=0}^{\infty} J_{k,m}^{(1)} \sigma^{-m/2} \tag{6.4}$$

analogous to (4.4) apply such that all J coefficients obey the relations (4.5). Then as in (5.5) we have the expansion

$$T_n(\sigma, x) = \sum_{m=0}^{\infty} B_{m,n}^{(1)}(x) \sigma^{-m/2} \tag{6.5a}$$

contingent on the definition

$$B_{m,n}^{(1)}(x) = \sum_{k=1}^{m} \hat{\alpha}_n^k(x) J_{k,m-k}^{(1)} . \tag{6.5b}$$

The convergence of (6.5a) is uniform over $0 \leq x \leq l$ and $\infty \geq |\sigma| \geq S_4$. Since the inverse Laplace transform of (6.2) is zero when $\tau = [T/l\sqrt{LC}] + 1$ (integer function) we do not inquire about the convergence of (6.2) with respect to n.

For $0 \leq x \leq l$ and $0 \leq n \leq \infty$, the σ domain step response of (6.2) with support from (5.3), (5.4) and (6.5) becomes

$$\begin{aligned} \hat{E}_2(\sigma, x) = \hat{E}_1(\sigma) \sum_{n=0}^{\infty}{}' /(-2)^n \epsilon^{-\hat{\alpha}_n(x) a_{1,2}^{(1)}} \Bigg\{ \sum_{k=0}^{\infty} \Bigg\{ \frac{1}{2} \sum_{m=0}^{k} A_{k+1-m,n-1}^{(1)} B_{m,n}^{(1)}(x) \Bigg\}_0 \Bigg\}_1 \\ \times \sigma^{-\left(\frac{k+n}{2}\right)} + \sum_{k=0}^{\infty} \Bigg\{ \sum_{m=0}^{k} A_{k+1-m,n}^{(1)} B_{m,n}^{(1)}(x) \Bigg\}_0 \sigma^{-\frac{k+n+1}{2}} \Bigg\}_1 \\ \times \exp\{-\hat{\alpha}_n(x)(\sigma + a_{1,1}^{(1)} \sigma^{1/2})\} \end{aligned} \tag{6.6}$$

where the polynomials of degree k are Sheffer and the prime on sigma indicates two exceptions:

(a) Replace $\sum_{m=0}^{k} A_{k+1-m,n-1}^{(1)} B_{m,n}^{(1)}(x)$ with $B_{k,n}^{(1)}(x)$ at $n=0$,

(b) Replace (6.6) at $n=0$ and $x=0$ with (5.3a).

Hence, the time domain response yields

$$\begin{aligned} \hat{e}_2(\tau, x) = E \sum_{n=0}^{[\tau]}{}' /(-2)^n \epsilon^{-\hat{\alpha}_n(x) a_{1,2}^{(1)}} \Bigg\{ \sum_{k=0}^{\infty} \Bigg\{ \frac{1}{2} \sum_{m=0}^{k} A_{k+1-m,n-1}^{(1)} B_{m,n}^{(1)}(x) \Bigg\}_0 \Bigg\}_1 \\ \times \Big\{ 4(\tau - \hat{\alpha}_n(x)) \Big\}_0^{\frac{k+n}{2}} I^{k+n} + \sum_{k=0}^{\infty} \Bigg\{ \sum_{m=0}^{k} A_{k+1-m,n}^{(1)} B_{m,n}^{(1)}(x) \Bigg\}_0 \\ \times \Big\{ 4(\tau - \hat{\alpha}_n(x)) \Big\}_0^{\frac{k+n+1}{2}} I^{k+n+1} \Bigg\}_1 \operatorname{erfc} \Bigg\{ \frac{\hat{\alpha}_n(x)}{2} a_{1,1}^{(1)} \Big\{ \tau - \hat{\alpha}_n(x) \Big\}_0^{-1/2} \Bigg\}_1 \end{aligned} \tag{6.7}$$

where (5.10) replaces (6.7) at $n=0$ and $x=0$.

To determine the τ interval of uniform convergence invoke the inequality

$$\{4(\tau-\hat{\alpha}_n(x))\}^{\frac{k+p}{2}} I^{k+p}\operatorname{erfc}\left\{\frac{\hat{\alpha}n(x)}{2}\, a_{1,1}^{(1)}\,(\tau-\hat{\alpha}_n(x))^{-1/2}\right\} \leqslant S_4^{-\frac{k+p}{2}} \tag{6.8}$$

for $p=n,\ n+1$.

For large time intervals we first note that (5.13) and (5.14) furnish a starting point for the expansion

$$1-\rho^2(s)\epsilon^{-2l\gamma(s)}=\frac{s^{1/2}}{(1+\eta_0 s^{1/2})^2}\sum_{n=0}^{\infty} M_n^{(3)} s^{n/2} \tag{6.9}$$

such that $M_n^{(3)}$ coefficients define according to the rules:

$$M_0^{(3)}=2\eta_0-b_{1,2}^{(3)}-A_{1,1}^{(3)},\qquad M_1^{(3)}-\eta_0^2-\{A_{2,1}^{(3)}+b_{2,2}^{(3)}+A_{1,1}^{(3)}b_{1,2}^{(3)}\},\qquad M_n^{(3)}=-\sum_{k=0}^{n+1} A_{n+1-k,1}^{(3)} b_{k,2}^{(3)},$$

where $A_{n,1}^{(3)}=\sum_{k=0}^{n} A_{n-k,0}^{(3)} A_{k,0}^{(3)}$.

From (5-12) and (6.9) we have the product

$$\frac{Z_0(s)}{Z_0(s)+\sqrt{\frac{L}{C}}}\,\frac{1}{1-\rho^2(s)\epsilon^{-2l\gamma(s)}}=\frac{(1+\eta_0 s^{1/2})\sum_{n=0}^{\infty} A_n^{(2)} s^{n/2}}{s^{1/2}\sum_{n=0}^{\infty} M_n^{(3)} s^{n/2}}. \tag{6.10}$$

Following the methods of (6.3) and (6.4) produces

$$\epsilon^{-\alpha_p(x)\gamma(s)}=\epsilon^{-\alpha_p(x)\sqrt{RC}s^{1/2}}\sum_{k=0}^{\infty}\frac{(\alpha_p(x)s)^k}{k!}\left\{\sum_{m=0}^{\infty} g_m^{(2)} s^{m/2}\right\}^k,$$

$$=\epsilon^{-\alpha_p(x)\sqrt{RC}s^{1/2}}\sum_{n=0}^{\infty} B_{n,p}^{(2)}(x)s^{n/2}, \tag{6.11a}$$

contingent on the following definitions and properties:

$$B_{n,p}^{(2)}(x)=\sum_{k=1}^{[n/2]}\frac{\alpha_p^k(x)}{k!}\,J_{k,n-2k}^{(2)}\qquad (n\geqslant 2), \tag{6.11b}$$

$$B_{0,p}^{(2)}(x)\equiv 1,\qquad B_{1,p}^{(2)}(x)\equiv 0,\qquad B_{n,0}^{(2)}(l)\equiv B_{n,1}^{(2)}(l). \tag{6.11c}$$

We define the equations

$$\left.\begin{aligned} C_{n,0}^{(2)}(x)&=\sum_{m=0}^{n} A_{n-m}^{(4)} B_{m,0}^{(2)}(x)\qquad (x>0),\\ C_{n,0}^{(2)}(0)&=A_n^{(4)}\end{aligned}\right\} \tag{6.12a}$$

$$A_0^{(4)}=1,\qquad A_n^{(4)}=A_n^{(2)}+\eta_0 A_{n-1}^{(2)}, \tag{6.12b}$$

$$C_{n,1}^{(2)}(x) = \sum_{k=0}^{n} \sum_{m=0}^{k} A_{n-k}^{(2)} A_{k-m,0}^{(3)} B_{m,1}^{(2)}(x), \tag{6.12c}$$

$$N_{n,k}(x) = \frac{C_{n,k}^{(2)}(x) - \sum_{m=0}^{n-1} N_{m,k}(x) M_{n-m}^{(3)}}{M_0^{(3)}} \qquad \begin{pmatrix} n \geq 1 \\ x \geq 0 \end{pmatrix}. \tag{6.12d}$$

From (6.11) and (6.12) three special cases condense into the forms

$$\left.\begin{aligned} N_{0,0}(x) &= N_{0,1}(x) = 1/M_0^{(3)}, \\ N_{1,0}(x) &= \frac{A_1^{(4)} - M_1^{(3)}/M_0^{(3)}}{M_0^{(3)}}, \\ N_{1,1}(x) &= \frac{A_1^{(2)} + A_{1,0}^{(3)} - M_1^{(3)}/M_0^{(3)}}{M_0^{(3)}}. \end{aligned}\right\} \quad x \geq 0. \tag{6.13}$$

In the s domain we have the representation

$$E_2(s, x) = \sum_{k=0}^{1} (-1)^k \epsilon^{-\alpha_k(x)\sqrt{RC}s^{1/2}} \sum_{n=0}^{\infty} N_{n,k}(x) s^{(n-3)/2} \tag{6.14}$$

which converges uniformly in x and s over $0 \leq x \leq l$ and $0 < \epsilon_0 \leq |s| \leq S_0$. It is necessary to consider three positions on the line: (a) $x=0$, (b) $x=l$, (c) $0<x<l$. At $x=0$ the terms corresponding to $n=0$ on Laplace transform inversion contribute $t^{1/2}$, a term which cancels on using the series expansion of I^1 erfc ( ). Hence, observe the complete asymptotic expansion for time domain reflectometry (TDR).

$$\begin{aligned} e_2(t, 0) \sim E \Bigg\{ & 2N_0 l\sqrt{RC} \sum_{m=1}^{\infty} \frac{(-)^{m-1} \left\{ \dfrac{2l\sqrt{RC}}{t^{1/2}} \right\}^{m-1}}{m!\Gamma\left(1 + \dfrac{1-m}{2}\right)} + N_{1,0}(0)U(t) \\ & + \sum_{n=1}^{\infty} (-1)^{n-1} N_{2n,0}(0) \Gamma\left(\tfrac{2n-1}{2}\right) t^{-\left(\frac{2n-1}{2}\right)} - N_{1,1}(0) \operatorname{erfc}\left\{ \frac{l\sqrt{RC}}{t^{1/2}} \right\} \\ & - \epsilon^{-\frac{l^2RC}{t}} \left\langle \sum_{n=2}^{\infty} N_{n,1}(0) \frac{H_{n-2}\left\{ \dfrac{l\sqrt{RC}}{t^{1/2}} \right\}}{2^{n-2} t^{(n-1)/2}} \right\rangle \Bigg\} \end{aligned} \tag{6.15}$$

as $t \to \infty$. In view of (6.12) and the coefficient definitions of section 5 (6.15) at $t = \infty$ reduces to

$$e_2(\infty, 0) = \frac{\sqrt{\dfrac{L}{C}} + lR}{2\sqrt{\dfrac{L}{C}} + lR}, \tag{6.16}$$

the d-c voltage divider equation.

Over $0 < x < l$ we use the expansion of I^1 erfc ( ) again to cancel the term associated with $t^{1/2}$ and extract the complete asymptotic expansion

$$e_2(t, x) \sim E\left\{N_0\sqrt{RC}\sum_{k=0}^{1}(-1)^k\alpha_k(x)\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m!\Gamma\left(1+\frac{1-m}{2}\right)}\left\{\frac{\alpha_k(x)\sqrt{RC}}{2t^{1/2}}\right\}^{m-1}\right.$$

$$\left.+\sum_{k=0}^{1}(-1)^k N_{1,k}\ \operatorname{erfc}\left(\frac{\alpha_k(x)\sqrt{RC}}{2t^{1/2}}\right)\right\}$$

$$+E\sum_{k=0}^{1}(-1)^k\epsilon^{-\frac{\alpha_k^2(x)RC}{4t}}\left\langle\sum_{n=2}^{\infty}N_{n,k}(x)\frac{H_{n-2}\left\{\frac{\alpha_k(x)\sqrt{RC}}{2t^{1/2}}\right\}}{2^{n-2}t^{(n-1)/2}}\right\rangle$$

as $t\to\infty$. In the case of time domain transmission (TDT) note $x=l$ means $\alpha_0(l)=\alpha_1(l)$ and, consequently the coefficient of N_0 vanishes. Also at $t=\infty$ and $x=l$ we obtain the d-c voltage divider expression

$$e_2(\infty, l)=\frac{\sqrt{\frac{L}{C}}}{2\sqrt{\frac{L}{C}}+lR}. \tag{6.18}$$

7. Realization of TDR and TDT Data

7.1. Theoretical and Experimental Results

A transmission line with the following parameter values is illustrated: $L\doteq 1.184\times 10^{-11}$ H/m, $C\doteq 2.46\times 10^{-7}$ F/m, $l\doteq 137.2$ m (450 ft), $R'\doteq 1.466\times 10^{-2}$ Ω/m, $K\sqrt{s}\doteq 3.35\times 10^{-5}\sqrt{s}$ Ω/m. Three curves of each figure correspond to three models of $Z(s)$ (sec. 1). Each theoretical curve approximates experimental data best over a certain time interval and the relative error between the two sets of data (taking either set as correct) over that time interval is realized through the following example:

TDT Data

t (microseconds)	1	2	3	4
% error	2	1	1	2

$$\Big\} Z(s)=\frac{R}{4}+K\sqrt{s}+Ls.$$

In view of a 3 percent uncertainty of the experimental system [6], a maximal 2 percent error for the appropriate $Z(s)$ in the data of the figures below is optimal. Experimental data points for t less than 10 ns are not reliable [6].

7.2. Computational Observations

Computer programs were written on a CDC 3800 digital computer in Fortran language to realize theoretical TDR and TDT data. We mention certain considerations of the computational procedure for (6.7) by starting with c and $d^{(1)}$ coefficients of (2.5a). Their product possesses recursions

$$F^{(1)}_{n+1,0}=\frac{n-\frac{1}{2}}{n+1}\beta_0 F^{(1)}_{n,0}, \qquad F^{(1)}_{n,k+1}=\frac{\left(n+k-\frac{1}{2}\right)(n-k)}{(2k+1)(2k+2)}\frac{\beta_2^2}{\beta_0}F^{(1)}_{n,k} \tag{7.2.1}$$

where

$$F^{(1)}_{n,k}=c_{n+k-1}d^{(1)}_{n+k,2k} \qquad (0\leq k\leq n \text{ and } n=1, 2, \ldots).$$

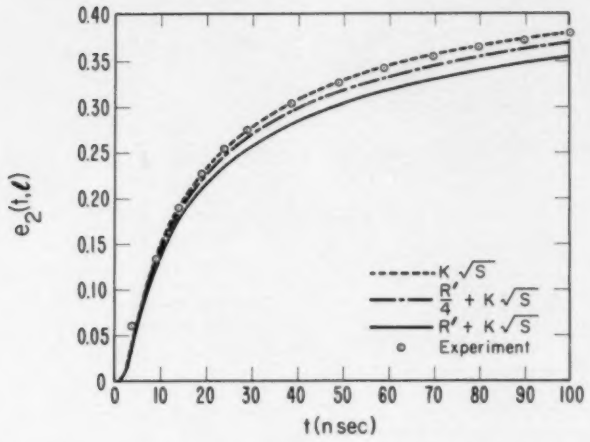

FIGURE 2. *Step response at receiving end with one unit of time delay* ($1\sqrt{LC}$) *removed.*

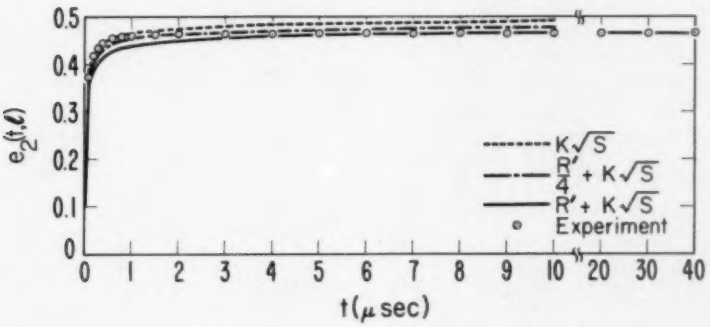

FIGURE 3. *Step response at receiving end with one unit of time delay* ($1\sqrt{LC}$) *removed.*

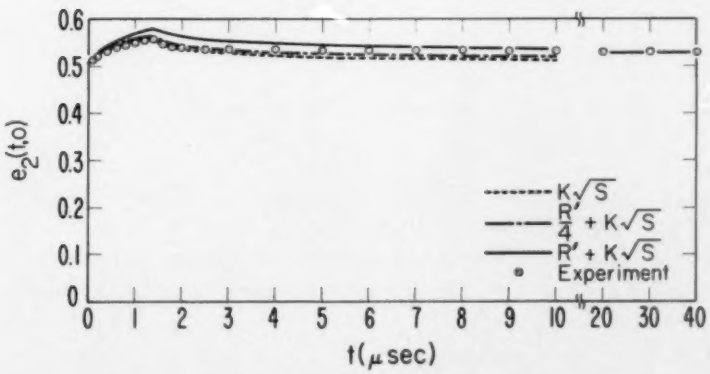

FIGURE 4. *Step response at sending end (no time delay).*

Similarly from (2.5b)

$$D^{(1)}_{n+1,0}=\beta_0\frac{n+\frac{1}{2}}{n+1}D^{(1)}_{n,0}, \qquad D^{(1)}_{n,k+1}=\frac{\left(n+k+\frac{1}{2}\right)(n-k)}{(2k+2)(2k+3)}\frac{\beta_2^2}{\beta_0}D^{(1)}_{n,k} \tag{7.2.2}$$

where

$$D^{(1)}_{n,k}=c_{n+k}d^{(1)}_{n+k+1,2k+1} \qquad (0\leq k\leq n \text{ and } n=0, 1, \ldots).$$

In computing $B^{(1)}_{m,n}(x)$ (at $x=0, l$) it makes no difference whether (4.5) or (5.5) is implemented to realize $J^{(1)}_{k,m-k}$ since indexing occurs over k. However, suppose inquiring *only* about the magnitude of reflection n occurs. Then coefficients $A^{(1)}_{m,n}$ compute via (4.5) by recurring over the mth now of the nth column [7] whereas (5.5) requires recursion between columns n and $n-1$.

For the time interval [8] $0\leq t\leq 10$ μs it is necessary to set $k=1(1)25$ and $n=1(1)10$ to compute the Sheffer polynomials

$$\sum_{m=0}^{k} A^{(1)}_{k+1-m,n}B^{(1)}_{m,n}(x).$$

Their ranges of magnitude at $x=0$ and $x=l$ appear below:

	$k=1$	$k=25$
$n=0$	$f_0\times 10^{-2}$	$f_1\times 10^{-14}$
$n=10$	$f_2\times 10^{-16}$	$f_3\times 10^{-23}$

$1<|f_i|<6 (i=0, \ldots, 3).$

Now determining numerical values of erfc u

$$(u=\frac{a^{(1)}_{1,1}}{2}\hat{\alpha}_n(x)\times(\tau-\hat{\alpha}_n(x))^{-1/2}$$

through its power series and using the recursion formula [16].

$$I^p \text{ erfc } u=-\frac{u}{p}I^{p-1} \text{ erfc } u+\frac{1}{2p}I^{p-2} \text{ erfc } u \tag{7.2.5}$$

enables term by term computation of (6.7). In reference to (6.7) let

$$\hat{e}_2(\tau, x)=\sum \hat{e}_{2,n}(\tau, x) \qquad \text{and} \qquad \hat{e}_{2,n}(\tau, x)=\sum \hat{e}_{2,n,k}(\tau, x).$$

Then a pair of practical terminating conditions for computing (6.7) can be stipulated:

(a) $|\hat{e}_{2,m,q(m)}(\tau, x)|\doteq 10^{-5}, \qquad m+q(m)\leq 25, \qquad m=1(1)p;$
(b) $|\hat{e}_{2,p}(\tau, x)|\doteq 10^{-4}, \qquad p\leq 10.$

To clarify these conditions suppose that $\hat{e}_2(\tau, 0)$ is wanted for $\tau=14.1$ ($t=10$ μs). Reference to (6.7) shows that $n=1(1)14$ while numerical computation reveals $\hat{e}_{2,m,q(m)}(\tau, 0)$ satisfies condition (a) for $m=1(1)4$. At $p=4$ conditions (a) and (b) are met and *all* computations on (6.7) are terminated.

A practical value for the interval of convergence may be determined by invoking the inequality

---

[7] Note the subscript position reversal of $A^{(1)}_{m,n}$ in comparing $C^{(1)}_{n,m}$ of (4.5).

[8] Setting $k=1(1)50$ and $n=1(1)10$ jumps computing time on the CDC 3800 from 30 s to 3 min. This happens because so many convolutions occur. However, it is not necessary to go beyond $k=25$ for this particular transmission line.

$$2^k(\tau-\hat{\alpha}_n(x))^{k/2} \quad I^k \operatorname{erfc}\left(\frac{a_{1,1}^{(1)}}{2}\,\hat{\alpha}_n(x)\,(\tau-\hat{\alpha}_n(x))^{-1/2}\right) \quad \leqslant 2^k\tau^{\frac{k}{2}}\frac{1}{\Gamma\left(\frac{k}{2}+1\right)} \tag{7.2.6}$$

Substituting the asymptotic expression $\Gamma(z) \sim 2\pi\epsilon^{-z}z^{z-1/2}$, $k=25$, and computing that value of τ which satisfies the equation

$$\tau=\{\Gamma(25/2+1)\}^{2/25}S_4^{-1} \tag{7.2.7}$$

produces $t \doteq 7.4\times10^{-6}$ seconds. Now $\hat{e}_{2,1,q(1)}(\tau, x)\,|_{x=0,\,t}$ does not satisfy condition (a) at $t=2.1\times10^{-5}$ s. Hence, (7.2.7) gives a pessimistic (but useful) guideline for the interval of convergence.

In asymptotic calculations terms corresponding to $N_{0,k}(x)$ and $N_{1,k}(x)$ are dominant and no more then three terms of each series (odd and even) are needed for this combination of parameter values. Asymptotic results are used for $t=20,\ 30,\ 40\ \mu$s.

8. Appendix A. Bound for $|\sigma|^{1/2}$

We have as a condition from the binomial series (2.1)

$$|\beta_0+\beta_2\sigma^{1/2}| < |\sigma|. \tag{A1}$$

Let $|\sigma|=v^2$ and invoke the triangular inequality to obtain [9]

$$v^2-|\beta_0+\beta_2\sigma^{1/2}| \geqslant v^2-\beta_2 v-\beta_0=f(v). \tag{A2}$$

Calculating the positive zero of $f(v)$ yields

$$v_0=\frac{\beta_2+\sqrt{\beta_2^2+4\beta_0}}{2}. \tag{A3}$$

It remains to show that $f(v)\geqslant 0$ for $v\geqslant v_0$. Noting $f(v)$ has one extremum, a minimum, at $v=\dfrac{\beta_2}{2}$ and $f'(v)\geqslant 0$ for $v\geqslant\beta_2/2$ completes the argument.

This bound is sharp since the selection $\sigma^{1/2}=v_0$ is an admissible value. Transforming from σ to s produces the result

$$S_2=\left\{\frac{K}{2L}+\sqrt{\frac{K}{4L^2}+\frac{R}{L}}\right\}^2. \tag{A4}$$

9. Appendix B. Asymptotic Expansion Discussion

The problem is to show that

$$e_2(t,\,l) \sim E \operatorname{erfc}\,(\delta t^{-1/2})+\frac{E}{\pi^{1/2}}\exp\left\{-\delta^2 t^{-1}\right\}\left\{\sum_{N=0}^{\infty} b_{2N+1,1}^{(2)}\right.$$

$$\left.\times\frac{H_{2n}(\delta t^{-1/2})}{2^{2n}t^{\frac{2n+1}{2}}}+b_{2N+2,1}^{(2)}\ +b_{2N+2,1}\,\frac{H_{2n+1}(\delta t^{-1/2})}{2^{2n+1}t^{n+1}}\right\} \qquad (t\to\infty) \tag{B1}$$

[9] The author is indebted to O. N. Strand for this argument.

is completely asymptotic $\left(\delta=\frac{l}{2}\sqrt{RC}\right)$.

Let $\varphi_n(t) \equiv H_{n-1}(\delta t^{-1/2})/t^{n/2}$ and define a sequence $\{\psi_n(t)\}_n$ such that

$$\lim_{t\to\infty}\frac{\psi_{n+1}(t)}{\psi_n(t)}=0 \qquad \text{for every } n. \tag{B2}$$

Then (B2) forms an asymptotic sequence [17] uniformly in n. Using the definition of Hermite polynomials [18] observe the limit quotient

$$\lim_{t\to\infty}\left|\frac{\varphi_{2n+1}(t)}{\varphi_{2n}(t)}\right|=\lim_{t\to\infty}\left|\frac{t^{-\frac{2n+1}{2}}}{t^{-\frac{2n+1}{2}}}\right|\frac{2}{\delta}=\frac{2}{\delta} \tag{B3}$$

as $t\to\infty$. Hence, a rearrangement is necessary to satisfy (B2).

It is quite easy to see that odd and even subsequences satisfy respectively

$$\lim_{t\to\infty}\left|\frac{\varphi_{2n+3}(t)}{\varphi_{2n+1}(t)}\right|=\lim_{t\to\infty}\left|\frac{\varphi_{2n+2}(t)}{\varphi_{2n}(t)}\right|=\lim_{t\to\infty}\mathrm{O}\left(\frac{1}{t}\right)=0. \tag{B4}$$

A complete asymptotic expansion with respect to $\{\psi_k(t)\}_k$ obeys the relation [19]

$$f(t)-\sum_{k=0}^{n-1}\psi_k(t)=\mathrm{O}\left(\psi_n(t)\right), \qquad t\to\infty. \tag{B5}$$

By definition of $\{\varphi_{2n}(t)\}_{2n}$, $|\varphi_{2N}(t)|\geq|\varphi_{2n}(t)|\,(n=N,\ N+1,\ \ldots)$ as $t\to\infty$. Also $b^{(2)}_{2n+1,1}$ origi- originates from a convergent series, i.e., $b^{(2)}_{2n+1,1}\leq M<\infty$. We have

$$\left|\sum_{n=N}^{\infty}b^{(2)}_{2n+1,1}\frac{\varphi_{2n}(t)}{2^{2n}}\right|\leq M\left|\varphi_{2N}(t)\right|\left|\sum_{n=N}^{\infty}\frac{1}{2^{2n}}=\mathrm{O}\ (\varphi_{2N}(t))\right., \tag{B6}$$

as $t\to\infty$, a relation which satisfies (B5). Clearly (6) also holds for the odd numbered terms. This argument extends to (6.15) and (6.17) in view of the fact $N_{n,k}(x)$ is uniformly bounded.

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Derivatives of the Grüneisen and Einstein Functions

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Expressions (in closed form) are derived for the first five derivatives of the Grüneisen and Einstein functions. Recursion formulas for the successive derivatives of both functions are also given. Computations are made and tabular results are given for the first five derivatives of the Grüneisen and Einstein functions.

Key words: Einstein function; electrical resistivity; Grüneisen function; solid state physics; transport properties.

1. Introduction

According to the solid state theory of matter, the mathematical expression for the electrical resistivity ρ, of good conductors is (ref. [1][1])

$$\rho = \frac{AT}{\theta}\left[G^0\left(\frac{\theta}{T}\right)\right] \tag{1}$$

where,

A = constant

θ = Debye temperature

T = absolute temperature

$$G^0\left(\frac{\theta}{T}\right) = 4\left(\frac{T}{\theta}\right)^4 \int_0^{\theta/T} \frac{\zeta^5 d\zeta}{(e^\zeta - 1)(1 - e^{-\zeta})}. \tag{2}$$

The function $G^0(\theta/T)$ is frequently referred to as the Grüneisen function. This function is approximately unity for $T \gg \theta$, which indicates that at high temperatures electrical resistivity increases linearly with temperature. At very low temperatures where $T \ll \theta$, the Grüneisen function can be expressed approximately by $G^0(\theta/T) \approx B(T/\theta)^4$, which indicates that at very low temperatures electrical resistivity increases with the fifth power of temperature.

For various calculations, the successive derivatives of the electrical resistivity function may be needed. This requires a knowledge of the derivatives of the Grüneisen function. Also, it may be possible to make further refinements in the electrical resistivity expression by expanding the resistivity function in the derivatives of the Grüneisen function. It was observed that the derivatives of the Grüneisen function contain the Einstein function and its successive derivatives. The objective of this writing is to obtain expressions for the successive derivatives of the Grüneisen and the Einstein functions.

[1] Figures in brackets indicate the literature references at the end of this paper.

2. Derivatives of the Grüneisen Function

Defining $x=\theta/T$, eq (2) becomes

$$G^0(x)=\frac{4}{x^4}\int_0^x \frac{\zeta^5 d\zeta}{(e^\zeta-1)(1-e^{-\zeta})} \tag{3}$$

Differentiating eq (3) and simplifying the result, one obtains

$$\frac{d}{dx}\left[G^0(x)\right]=G^1(x)=-\frac{4}{x}\left[\frac{4}{x^4}\int_0^x \frac{\zeta^5 d\zeta}{(e^\zeta-1)\,(1-e^{-\zeta})}\right]+\frac{4}{x}\left[\frac{x^2e^{-x}}{(1-e^{-x})^2}\right]. \tag{4}$$

The Einstein function is defined as

$$E^0(x)=\frac{x^2e^{-x}}{(1-e^{-x})^2}. \tag{5}$$

Substitution of eqs (3) and (5) in eq (4) yields

$$G^1(x)=-\frac{4}{x}\left[\,G^0(x)-E^0(x)\,\right]. \tag{6}$$

For convenience, the variable x in the functions $G^0(x)$, $E^0(x)$, and in their derivatives $G^1(x)$, $E^1(x)$, etc. will be omitted and will be written simply as G^0, E^0, G^1, E^1, etc., respectively. One may note that the superscript numerals on both G and E indicate the order of differentiation.

The following is a list of the first five derivatives of the Grüneisen function obtained by the successive differentiation of eq (3) expressed in terms of the lower order derivatives of the Grüneisen and Einstein functions.

$$G^1=-\frac{4}{x}G^0+\frac{4}{x}E^0 \tag{7}$$

$$G^2=-\frac{5}{x}G^1+\frac{4}{x}E^1 \tag{8}$$

$$G^3=-\frac{6}{x}G^2+\frac{4}{x}E^2 \tag{9}$$

$$G^4=-\frac{7}{x}G^3+\frac{4}{x}E^3 \tag{10}$$

$$G^5=-\frac{8}{x}G^4+\frac{4}{x}E^4 \tag{11}$$

The recursion formula for the above is

$$G^n=-\left(\frac{n+3}{x}\right)G^{n-1}+\frac{4}{x}E^{n-1} \tag{12}$$

Where, n is the order of differentiation. It may be seen that the nth derivative of the Grüneisen function is given in terms of its $(n-1)$th derivative. In order to be able to express the nth derivative of the Grüneisen function in terms of the function itself, successive substitutions for the lower order derivatives are made.

The following is a list of the first five derivatives of the Grüneisen function in terms of the Grüneisen function and the derivatives of the Einstein function.

$$G^1 = -\frac{4}{x}(G^0 - E^0) \tag{13}$$

$$G^2 = \frac{20}{x^2}(G^0 - E^0) + \frac{4}{x}E^1 \tag{14}$$

$$G^3 = -\frac{120}{x^3}(G^0 - E^0) - \frac{24}{x^2}E^1 + \frac{4}{x}E^2 \tag{15}$$

$$G^4 = \frac{840}{x^4}(G^0 - E^0) + \frac{168}{x^3}E^1 - \frac{28}{x^2}E^2 + \frac{4}{x}E^3 \tag{16}$$

$$G^5 = -\frac{6720}{x^5}(G^0 - E^0) - \frac{1344}{x^4}E^1 + \frac{224}{x^3}E^2 - \frac{32}{x^2}E^3 + \frac{4}{x}E^4 \tag{17}$$

The recursion formula for the above is

$$G^n = \frac{S_{n1}C_{n1}}{x^n}(G^0 - E^0) + \frac{S_{n2}C_{n2}}{x^{n-1}}E^1 + \frac{S_{n3}C_{n3}}{x^{n-2}}E^2 + \ldots + \frac{S_{ni}C_{ni}}{x^{n-i+1}}E^{i-1} + \ldots + \frac{S_{nn}C_{nn}}{x}E^{n-1} \tag{18}$$

where, C_{ni} and S_{ni} indicate the absolute value and the sign of the coefficients, respectively, and are expressed by the following relations.

$$\left.\begin{aligned} C_{11} &= 4 \\ C_{ni} &= 0 && [i > n] \\ C_{n1} &= (n+3)\ C_{n-1,1} && [n \geq 2] \\ C_{ni} &= (n-i)\ C_{n-1,i} + C_{n-1,i-1} && [n \geq i \geq 2] \\ S_{n1} &= (-1)^n \\ S_{ni} &= (-1)^{n+i} && [n \geq i \geq 2] \end{aligned}\right\} \tag{19}$$

The numerical values of the product $S_{ni}C_{ni}$ required for the first seven derivatives of the Grüneisen function are given in table 1.

It can be seen from eq (18) that the relation for the derivative of the Grüneisen function contains the derivatives of the Einstein function. The latter are obtained in the next section.

3. Derivatives of the Einstein Function

The Einstein function is defined by eq (5). For mathematical convenience, it may be written as

$$E^0 = \left(\frac{x^2}{1-e^{-x}}\right) y \tag{20}$$

TABLE 1. *List of the coefficients* $S_{ni}C_{ni}$ *that appear in eq (18)*

	$S_{ni}C_{ni}$	$i\rightarrow$ 1	2	3	4	5	6	7
	1	−4	0	0	0	0	0	0
	2	20	4	0	0	0	0	0
n	3	−120	−24	4	0	0	0	0
↓	4	840	168	−28	4	0	0	0
	5	−6,720	−1,344	224	−32	4	0	0
	6	60,480	12,096	−2,016	288	−36	4	0
	7	−604,800	−120,960	20,160	−2,880	360	−40	4

where,

$$y=\frac{e^{-x}}{1-e^{-x}}=\frac{1}{e^{x}-1}. \tag{21}$$

Differentiation of eq (20) and simplification yields

$$\frac{dE^{0}}{dx}=E^{1}=2D^{0}E^{0} \tag{22}$$

where,

$$D^{0}=-\frac{1}{2}+\frac{1}{x}-y. \tag{23}$$

The second derivative of E^0 becomes

$$\frac{dE^{1}}{dx}=E^{2}=2D^{0}E^{1}+2D^{1}E^{0} \tag{24}$$

where,

$$\mathrm{D}^{1}=\frac{dD^{0}}{dx}=-\frac{1}{x^{2}}+y+y^{2}. \tag{25}$$

It should be noted that the superscript numeral of D indicates the order of differentiation similar to those used for G and E. It may be seen that the derivatives of the Einstein function contain the derivatives of the quantity y^m for various values of m, where m indicates the power of y. The recursion formula for the derivatives of y^m is

$$\frac{d}{dx}\left(y^{m}\right)=-my^{m}(y+1). \tag{26}$$

The following is a list of the first five derivatives of the Einstein function obtained by the successive differentiation of eq (20).

$$E^{1}=2D^{0}E^{0} \tag{27}$$

$$E^{2}=2D^{0}E^{1}+2D^{1}E^{0} \tag{28}$$

$$E^{3}=2D^{0}E^{2}+4D^{1}E^{1}+2D^{2}E^{0} \tag{29}$$

$$E^4 = 2D^0E^3 + 6D^1E^2 + 6D^2E^1 + 2D^3E^0 \tag{30}$$

$$E^5 = 2D^0E^4 + 8D^1E^3 + 12D^2E^2 + 8D^3E^1 + 2D^4E^0 \tag{31}$$

The recursion formula for the above is

$$E^n = K_{n1}D^0E^{n-1} + K_{n2}D^1E^{n-2} + K_{n3}D^2E^{n-3} + \ldots + K_{ni}D^{m-1}E^{n-i} + \ldots + K_{nn}D^{n-1}E^0 \tag{32}$$

where, n indicates the order of differentiation.

K_{ni} are constants and can be obtained from the following relations

$$\left.\begin{aligned} &K_{n1} = 2 \\ &K_{ni} = 0 \qquad\qquad [i > n] \\ &K_{ni} = K_{n-1,i} + K_{n-1,i-1} \qquad [n \geq i \geq 2] \end{aligned}\right\} \tag{33}$$

D^m are coefficients which are functions of x. The superscript m indicates the order of differentiation. They are expressed by the following equations.

$$D^0 = -\frac{1}{2} + \frac{1}{x} - y \tag{34}$$

$$D^m = (-1)^{m+1}\left[-\frac{m!}{x^{m+1}} + \kappa_{m1}y + \kappa_{m2}y^2 + \kappa_{m3}y^3 + \ldots + \kappa_{mj}y^j + \ldots + \kappa_{m,m+1}y^{m+1}\right]_{[m \geq 1]} \tag{35}$$

where,

$$\left.\begin{aligned} &\kappa_{m1} = \kappa_{12} = 1 \\ &\kappa_{mj} = 0 \qquad\qquad [j > (m+1)] \\ &\kappa_{mj} = j(\kappa_{m-1,j}) + (j-1)(\kappa_{m-1,j-1}) \qquad [j \geq 2,\ m \geq 2,\ j \leq (m+1)]. \end{aligned}\right\} \tag{36}$$

The coefficients K_{ni} and κ_{mj} can be represented as matrix elements. Their numerical values, required for the first seven derivatives of the Einstein function, are given in tables 2 and 3, respectively.

TABLE 2. *List of the coefficients* K_{ni} *that appear in eq (32)*

K_{ni}	$i \rightarrow$ 1	2	3	4	5	6	7
1	2	0	0	0	0	0	0
2	2	2	0	0	0	0	0
n 3	2	4	2	0	0	0	0
↓ 4	2	6	6	2	0	0	0
5	2	8	12	8	2	0	0
6	2	10	20	20	10	2	0
7	2	12	30	40	30	12	2

TABLE 3. *List of the coefficients* κ_{mj} *that appear in eq (35)*

κ_{mj}	$j \rightarrow$ 1	2	3	4	5	6	7	8
1	1	1	0	0	0	0	0	0
2	1	3	2	0	0	0	0	0
m 3	1	7	12	6	0	0	0	0
↓ 4	1	15	50	60	24	0	0	0
5	1	31	180	390	360	120	0	0
6	1	63	602	2,100	3,360	2,520	720	0
7	1	127	1,932	10,206	25,200	31,920	20,160	5,040

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Calculating the derivatives of the Einstein function using eq (32) and substituting these in either eq (12) or eq (18) one can obtain the derivatives of the Grüneisen function. Since the nth derivative of the Grüneisen function contains all lower derivatives (up to $n-1$) of the Einstein function, it is sufficient to derive a recursion formula for the derivatives of the Einstein function in terms of its lower derivatives. However, in the case of the Grüneisen function, a recursion formula for its derivatives was obtained both in terms of lower derivatives of G and in terms of G itself.

4. Computation of the Derivatives of the Einstein and Grüneisen Functions

Based on the formulas developed in the previous sections, the first five derivatives of the Einstein and Grüneisen functions are computed for the range $1 < x < 30$ using a computer. The selection of the limits is that of a practical one. Above the Debye temperature ($x = 1$) both functions are relatively insensitive to temperature variations and approach unity as temperature increases. Below $T = \theta/30 (x = 30)$ the values of the functions become small and also the integral in the Grüneisen function attains a constant value.

The Einstein function and its first five derivatives are computed using eqs (20), (27), (28), (29), (30), and (31), respectively in conjunction with eqs (34), (35), and (36). The results are given in table 4. The values obtained for the Einstein function are in exact agreement with those reported by Hilsenrath and Ziegler [3].

TABLE 4. *The Einstein function and its first five derivatives**

X	E^0		E^1		E^2		E^3		E^4		E^5	
	Coef	Exp	Coef	Exp	Coef	Exp	Coef	Exp	Coef	Exp	Coef	Exp
1	9.20674	−1	−1.50948	−1	−1.21319	−1	8.19752	−2	4.93166	−2	−8.53277	−2
2	7.24062	−1	−2.26657	−1	−2.89466	−2	8.94460	−2	−2.89868	−2	−5.50503	−2
3	4.96269	−1	−2.17428	−1	3.97080	−2	4.43374	−2	−5.06061	−2	7.34212	−3
4	3.04087	−1	−1.63391	−1	6.13399	−2	2.48308	−3	−3.00279	−2	2.62514	−2
5	1.70742	−1	−1.04762	−1	5.29512	−2	−1.54893	−2	−7.59110	−3	1.66883	−2
6	8.96791	−2	−6.02318	−2	3.59185	−2	−1.68202	−2	3.02914	−3	5.38369	−3
7	4.47638	−2	−3.20559	−2	2.12103	−2	−1.22491	−2	5.20388	−3	−1.14766	−4
8	2.14840	−2	−1.61274	−2	1.14494	−2	−7.45505	−3	4.15200	−3	−1.55181	−3
9	9.99866	−3	−7.77921	−3	5.80800	−3	−4.08606	−3	2.61503	−3	−1.39754	−3
10	4.54041	−3	−3.63274	−3	2.81612	−3	−2.09076	−3	1.45695	−3	−9.15259	−4
11	2.02097	−3	−1.65359	−3	1.31966	−3	−1.01920	−3	7.52291	−4	−5.19021	−4
12	8.84777	−4	−7.37325	−4	6.02169	−4	−4.79315	−4	3.68772	−4	−2.70560	−4
13	3.81997	−4	−3.23230	−4	2.68985	−4	−2.19263	−4	1.74065	−4	−1.33395	−4
14	1.62980	−4	−1.39697	−4	1.18078	−4	−9.81221	−5	7.98299	−5	−6.32020	−5
15	6.88281	−5	−5.96510	−5	5.10858	−5	−4.31325	−5	3.57911	−5	−2.90616	−5
16	2.88090	−5	−2.52079	−5	2.18318	−5	−1.86809	−5	1.57550	−5	−1.30542	−5
17	1.19644	−5	−1.05568	−5	9.23206	−6	−7.99009	−6	6.83091	−6	−5.75454	−6
18	4.93451	−6	−4.38623	−6	3.86842	−6	−3.38106	−6	2.92416	−6	−2.49772	−6
19	2.02261	−6	−1.80970	−6	1.60800	−6	−1.41751	−6	1.23822	−6	−1.07013	−6
20	8.24461	−7	−7.42015	−7	6.63691	−7	−5.89490	−7	5.19411	−7	−4.53454	−7
21	3.34391	−7	−3.02544	−7	2.72214	−7	−2.43400	−7	2.16103	−7	−1.90322	−7
22	1.35010	−7	−1.22737	−7	1.11021	−7	−9.98630	−8	8.92630	−8	−7.92209	−8
23	5.42853	−8	−4.95649	−8	4.50497	−8	−4.07397	−8	3.66349	−8	−3.27354	−8
24	2.17448	−8	−1.99327	−8	1.81961	−8	−1.65351	−8	1.49495	−8	−1.34395	−8
25	8.67997	−9	−7.98557	−9	7.31895	−9	−6.68010	−9	6.06903	−9	−5.48574	−9
26	3.45374	−9	−3.18807	−9	2.93262	−9	−2.68738	−9	2.45236	−9	−2.22756	−9
27	1.37018	−9	−1.26868	−9	1.17095	−9	−1.07697	−9	9.86753	−10	−9.00294	−10
28	5.42089	−10	−5.03368	−10	4.66031	−10	−4.30076	−10	3.95504	−10	−3.62315	−10
29	2.13922	−10	−1.99169	−10	1.84925	−10	−1.71189	−10	1.57962	−10	−1.45243	−10
30	8.42186	−11	−7.86040	−11	7.31766	−11	−6.79363	−11	6.28832	−11	−5.80173	−11

* The superscript numerals indicate the order of differentiation.

The Grüneisen function and its first five derivatives are computed using eqs (3), (13), (14), (15), (16), and (17), respectively. The results are given in table 5. The Grüneisen function expressed

TABLE 5. *The Grüneisen function and its first five derivatives**

X	J_5^{**}		G^0		G^1		G^2		G^3		G^4		G^5	
	Coef	Exp	Coef	Exp	Coef	Exp	Coef	Exp	Coef	Exp	Coef	Exp	Coef	Exp
1	2.36616	−1	9.46463	−1	−1.03156	−1	−8.80120	−2	4.27961	−2	2.83281	−2	−2.93585	−2
2	3.22929	0	8.07321	−1	−1.66518	−1	−3.70190	−2	5.31638	−2	−7.18128	−3	−2.92485	−2
3	1.27705	+1	6.30643	−1	−1.79165	−1	8.70489	−3	3.55342	−2	−2.37967	−2	−4.01706	−3
4	2.94882	+1	4.60754	−1	−1.56667	−1	3.24428	−2	1.26758	−2	−1.96995	−2	9.37115	−3
5	5.02629	+1	3.21682	−1	−1.20752	−1	3.69424	−2	−1.96992	−3	−9.63355	−3	9.34080	−3
6	7.08729	+1	2.18744	−1	−8.60433	−2	3.15482	−2	−7.60252	−3	−2.34386	−3	5.14457	−3
7	8.83248	+1	1.47147	−1	−5.85047	−2	2.34714	−2	−7.99817	−3	9.98689	−4	1.83229	−3
8	1.01477	+2	9.90984	−2	−3.88072	−2	1.61908	−2	−6.41840	−3	1.88858	−3	1.87425	−4
9	1.10546	+2	6.73958	−2	−2.55098	−2	1.07147	−2	−4.56180	−3	1.73204	−3	−3.77359	−4
10	1.16379	+2	4.65517	−2	−1.68045	−2	6.94916	−3	−3.04305	−3	1.29383	−3	−4.52284	−4
11	1.19927	+2	3.27647	−2	−1.11795	−2	4.48030	−3	−1.96393	−3	8.79152	−4	−3.65823	−4
12	1.21988	+2	2.35317	−2	−7.54897	−3	2.89963	−3	−1.24909	−3	5.68866	−4	−2.56320	−4
13	1.23141	+2	1.72460	−2	−5.18892	−3	1.89628	−3	−7.92444	−4	3.59235	−4	−1.67509	−4
14	1.23765	+2	1.28868	−2	−3.63538	−3	1.25844	−3	−5.05593	−4	2.24762	−4	−1.05627	−4
15	1.24093	+2	9.80489	−3	−2.59628	−3	8.49521	−4	−3.26185	−4	1.40718	−4	−6.55052	−5
16	1.24262	+2	7.58434	−3	−1.88888	−3	5.83974	−4	−2.13532	−4	8.87501	−5	−4.04363	−5
17	1.24347	+2	5.95523	−3	−1.39842	−3	4.08815	−4	−1.42115	−4	5.66380	−5	−2.50459	−5
18	1.24388	+2	4.73969	−3	−1.05217	−3	2.91294	−4	−9.62384	−5	3.66747	−5	−1.56501	−5
19	1.24409	+2	3.81853	−3	−8.03475	−4	2.11060	−4	−6.63120	−5	2.41323	−5	−9.90029	−6
20	1.24418	+2	3.11045	−3	−6.21925	−4	1.55333	−4	−4.64671	−5	1.61456	−5	−6.35436	−6
21	1.24422	+2	2.55907	−3	−4.87378	−4	1.15985	−4	−3.30867	−5	1.09825	−5	−4.14266	−6
22	1.24424	+2	2.12459	−3	−3.86265	−4	8.77651	−5	−2.39157	−5	7.59140	−6	−2.74428	−6
23	1.24425	+2	1.77851	−3	−3.09297	−4	6.72298	−5	−1.75304	−5	5.32824	−6	−1.84693	−6
24	1.24425	+2	1.50011	−3	−2.50015	−4	5.20831	−5	−1.30177	−5	3.79408	−6	−1.26220	−6
25	1.24425	+2	1.27411	−3	−2.03856	−4	4.07700	−5	−9.78362	−6	2.73834	−6	−8.75299	−7
26	1.24425	+2	1.08912	−3	−1.67556	−4	3.22219	−5	−7.43537	−6	2.00142	−6	−6.15443	−7
27	1.24425	+2	9.36511	−4	−1.38742	−4	2.56928	−5	−5.70934	−6	1.48004	−6	−4.38384	−7
28	1.24425	+2	8.09722	−4	−1.15674	−4	2.06561	−5	−4.42624	−6	1.10650	−6	−3.16086	−7
29	1.24425	+2	7.03681	−4	−9.70594	−5	1.67344	−5	−3.46225	−6	8.35693	−7	−2.30514	−7
30	1.24425	+2	6.14445	−4	−8.19260	−5	1.36543	−5	−2.73085	−6	6.37190	−7	−1.69909	−7

*The superscript numerals indicate the order of differentiation.
** For $x=\infty$, $J_5=124.4313$ [2].

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by eq (3) may be written as

$$G^0(x) = \left(\frac{4}{x^4}\right) J_5 \tag{37}$$

where

$$J_5 = \int_0^x \frac{\zeta^5 d\zeta}{(e^\zeta - 1)(1 - e^{-\zeta})}. \tag{38}$$

There is no known closed form for the above integration, therefore numerical integration techniques are employed. Various values for the increment in numerical integration are tried to determine the convergence of the results. Values 0.002 and 0.0002 yielded a difference of 0.00042 percent for $x = 1$, and 0.0048 percent for $x = 30$ in J_5. In the literature, Rogers and Powell [4] have computed the integral J_5 utilizing series expansion approximation. Their results correspond exactly to those obtained using a value of 0.002 for the increment. Since the accuracy of the approximation of numerical integration increases with decreasing increment size, a value of 0.0002 is used in the final computations. Thus, it is expected that the present results of J_5 and hence the Grüneisen function are more accurate than those given by Rogers and Powell.

It may be noted that the only numerical method employed in the present computations is that for the evaluation of J_5, the integral appearing in the Grüneisen function. All other computations including those for the first five derivatives of the Grüneisen and Einstein functions represent mathematically closed forms and thus are not subject to errors common to numerical differentiation.

5. Summary

The Grüneisen and Einstein functions appear in the mathematical expressions describing a number of thermodynamic and transport properties of matter. In order to improve the interpretation of experimental results and extend the application of the theories regarding these properties, a knowledge of the temperature derivatives of the above functions are needed. In the present study, mathematical expressions in closed form are derived for the derivatives of the Grüneisen and Einstein functions. Also, the first five derivatives of both functions are computed and presented in tabular form. Since all the derivatives are obtained from mathematical relations of closed form, no errors inherent to numerical differentiation are introduced.

The author extends his appreciation to C. W. Beckett for suggesting the undertaking of this study.

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Selecting Nonlinear Transformations for the Evaluation of Improper integrals*

T. A. Atchison**

(May 25, 1970)

Recent literature concerning the use of nonlinear transformations to evaluate numerically certain improper integrals of the first kind involves the determination of a transformation function g to improve the approximation. By approximating a given integrand f by an integrable function f_1 and then determining an associated g function for f_1, a nonlinear transformation may be constructed which will yield an improved approximation of the improper integral of f.

Key words: Improper integral, nonlinear transformation.

1. Introduction

H. L. Gray and T. A. Atchison[1] introduced a class of nonlinear transformations to assist in the numerical evaluation of improper integrals of the first kind. A transformation from this class is completely determined by specifying the integrand of the improper integral and a transformation function g satisfying certain mild restrictions. The purpose of this paper is to exhibit a scheme for the selection of g which will yield a good approximation to the improper integral.

2. Selecting the Transformation

The Generalized G-transform of Atchison and Gray is as follows:
If f is continuous on $[a, \infty)$ and

$$F(t)=\int_a^t f(x)dx \to S \text{ as } t \to \infty, \tag{2.1}$$

then

$$G[F; g; t]=\frac{F(t)-R(t)F(g(t))}{1-R(t)} \tag{2.2}$$

where $g \epsilon C^1$ and $g(t) \geq a$ on $[a, \infty)$, $\lim_{t\to\infty} g(t)=\infty$, and

$$R(t)=\frac{f(t)}{f(g(t))g'(t)} \tag{2.3}$$

One of the basic theorems proved (see footnote 1) concerns the determination of a differentiable function g such that $G[F; g; t] \equiv S$ for all $t \geq t_0 \geq a$ for some class of functions f:
THEOREM: *A necessary and sufficient condition that* $G[F; g; t] \equiv S$ *for all* $t \geq t_0 \geq a$ *is that* $R(t)$ *is constant for* $t \geq t_0$.

In general, for a given integrand f, the determination of a function g for which $R(t)$ is constant is difficult. However, suppose f_1 is an integrable function which approximates f. Requiring the

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[1] H. L. Gray and T. A. Atchison, The Generalized G-transformation, Math. Comp. 22, No. 103, 595–606 (1968).

corresponding $R(t)$ to be constant we get

$$f_1(g(t))g'(t)=c_1f_1(t) \qquad \text{for } t \geqslant t_0. \tag{2.4}$$

If $y=g(t)$, then (2.4) becomes

$$f_1(y)dy=c_1f_1(t)dt \tag{2.5}$$

and integration yields

$$\int f_1(y)dy=c_1\int f_1(t)dt+c_2, \tag{2.6}$$

where c_1 and c_2 are constants. Thus, if

$$H(y)=\int f_1(y)dy, \tag{2.7}$$

then any differentiable function g which satisfies the functional equation

$$H(y)=c_1H(t)+c_2 \qquad \text{for } t \geqslant t_0. \tag{2.8}$$

will result in

$$G[F_1;\ g;\ t] \equiv S_1 \qquad \text{for } t \geqslant t_0. \tag{2.9}$$

where

$$F_1(t)=\int_a^t f_1(x)dx \to S_1 \text{ as } t \to \infty. \tag{2.10}$$

Since f_1 approximates f, then $G[F_1;\ g;\ t]$ will be an approximation of S.

To illustrate this procedure, consider

$$F(t)=\frac{2}{\sqrt{\pi}}\int_a^t e^{-x^2}dx \to \operatorname{erfc}(a) \qquad \text{as } t \to \infty, \tag{2.11}$$

where $0<a<\infty$. For $y \geqslant 1$, the integrand of (2.11) may be approximated by

$$f_1(y)=ye^{-y^2} \tag{2.12}$$

and f_1 possesses an integral

$$H(y)=-\frac{1}{2}e^{-y^2}. \tag{2.13}$$

One may determine a function $y=g(t)$ which satisfies (2.8) by choosing $c_1=e^{-k^2}$ and $c_2=0$. Such a function is

$$g(t)=\sqrt{t^2+k^2}. \tag{2.14}$$

Applying (2.2) to (2.11) and utilizing (2.14) we get

$$G[F; g; t] = \frac{F(t) - e^{k^2}\frac{\sqrt{t^2+k^2}}{t} F(\sqrt{t^2+k^2})}{1 - e^{k^2}\frac{\sqrt{t^2+k^2}}{t}} \tag{2.15}$$

Theorem 1 (see footnote 1) may be applied to (2.15) to show that it converges to erfc (a) more rapidly than either $F(t)$ on $F(\sqrt{t^2+k^2})$. This is illustrated by the following numerical values in the case $a=1$, $k=1$:

t	$F(t)$	Error	$G[F; g; t]$	Error
1.0	0.0	0.15729921	0.15023133	0.00706788
1.5	.12340436	.03389485	.15655151	.00074770
2.0	.15262147	.0046777	.15745495	.00015574

If we apply Aitken's Δ^2-process to the $F(t)$ column [2] the resulting approximation is 0.16168462 which is in error by 0.00438541.

A general procedure for solving the functional equation (2.8) is not available. Also, for different choices of c_1 and c_2, other solutions of the functional equation may be determined.

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On the Spheroidal Functions *

Donald R. Rhodes **

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A number of new properties of the spheroidal functions of arbitrary real order $\alpha > -1$ are established, including double orthogonality over two separate intervals simultaneously and the existence of a new kind of characteristic numers $\gamma_{\alpha n}(c)$ that arise from it. Some computational formulas are derived and a few numerical results are shown.

Key words: Mathieu functions; spheroidal functions.

1. Introduction

The spheroidal functions as defined and investigated by Stratton [1] [1] and later by Chu and Stratton [2] are those solutions of the differential equation

$$(1-\eta^2)\psi''_{\alpha n}(c, \eta) - 2(\alpha+1)\eta\psi'_{\alpha n}(c, \eta) + (b_{\alpha n} - c^2\eta^2)\psi_{\alpha n}(c, \eta) = 0 \tag{1}$$

that remain finite at the singular points $\eta = \pm 1$. The condition of finiteness restricts the admissible values of the parameter $b_{\alpha n}(c)$ to a discrete set of eigenvalues, indexed by $n = 0, 1, 2, 3, \ldots$, that depend upon the values chosen for the order α and the parameter c. The differential equation (1) together with the condition of finiteness at $\eta = \pm 1$ is equivalent to the integral equation [1(27)]

$$\nu_{\alpha n}(c)\,\psi_{\alpha n}(c, \eta) = \int_{-1}^{1} e^{ic\eta t}(1-t^2)^{\alpha}\psi_{\alpha n}(c, t)\,dt, \quad -1 \leq \eta \leq 1, \tag{2}$$

with eigenvalues $\nu_{\alpha n}(c)$, valid for real $\alpha > -1$.

For integral orders $\alpha = m = 0, 1, 2, 3, \ldots$ the functions $\psi_{\alpha n}(c, \eta)$ are related to the spheroidal wave functions $S_{m, m+n}(c, \eta)$ that arise from separation of the wave equation in spheroidal coordinates. For half-integral orders $\alpha = \pm\frac{1}{2}$ they are related to the periodic Mathieu functions $Se_n(c, \eta)$ and $So_{n+1}(c, \eta)$ that arise from separation of the wave equation in elliptic cylinder coordinates. The purpose of the present paper is to derive some new properties of the spheroidal functions of arbitrary real order $\alpha > -1$, including a new kind of characteristic numbers $\gamma_{\alpha n}(c)$ that arise from the property of double orthogonality shown here to be possessed by these functions, and to present some numerical results illustrating them. It will be assumed throughout that the parameter c is real and positive, although many of the results are more general. A secondary purpose is to emphasize the underlying unity of the properties associated with the spheroidal functions of arbitrary real order $\alpha > -1$. All properties of the spheroidal wave functions and of the periodic Mathieu functions arise quite naturally from those of the general functions $\psi_{\alpha n}(c, \eta)$. An extensive and completely general treatment for all solutions of (1) for arbitrary complex values of α, b, and c has been developed by Meixner and Schäfke [3].

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[1] Figures in brackets indicate the literature references at the end of this paper.

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On the Spheroidal Functions *

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A number of new properties of the spheroidal functions of arbitrary real order $\alpha > -1$ are established, including double orthogonality over two separate intervals simultaneously and the existence of a new kind of characteristic numers $\gamma_{\alpha n}(c)$ that arise from it. Some computational formulas are derived and a few numerical results are shown.

Key words: Mathieu functions; spheroidal functions.

1. Introduction

The spheroidal functions as defined and investigated by Stratton [1] [1] and later by Chu and Stratton [2] are those solutions of the differential equation

$$(1-\eta^2)\psi''_{\alpha n}(c, \eta) - 2(\alpha+1)\eta\psi'_{\alpha n}(c, \eta) + (b_{\alpha n} - c^2\eta^2)\psi_{\alpha n}(c, \eta) = 0 \tag{1}$$

that remain finite at the singular points $\eta = \pm 1$. The condition of finiteness restricts the admissible values of the parameter $b_{\alpha n}(c)$ to a discrete set of eigenvalues, indexed by $n = 0, 1, 2, 3, \ldots$, that depend upon the values chosen for the order α and the parameter c. The differential equation (1) together with the condition of finiteness at $\eta = \pm 1$ is equivalent to the integral equation [1(27)]

$$\nu_{\alpha n}(c)\,\psi_{\alpha n}(c, \eta) = \int_{-1}^{1} e^{ic\eta t}(1-t^2)^{\alpha}\psi_{\alpha n}(c, t)dt, \quad -1 \leq \eta \leq 1, \tag{2}$$

with eigenvalues $\nu_{\alpha n}(c)$, valid for real $\alpha > -1$.

For integral orders $\alpha = m = 0, 1, 2, 3, \ldots$ the functions $\psi_{\alpha n}(c, \eta)$ are related to the spheroidal wave functions $S_{m, m+n}(c, \eta)$ that arise from separation of the wave equation in spheroidal coordinates. For half-integral orders $\alpha = \pm\frac{1}{2}$ they are related to the periodic Mathieu functions $Se_n(c, \eta)$ and $So_{n+1}(c, \eta)$ that arise from separation of the wave equation in elliptic cylinder coordinates. The purpose of the present paper is to derive some new properties of the spheroidal functions of arbitrary real order $\alpha > -1$, including a new kind of characteristic numbers $\gamma_{\alpha n}(c)$ that arise from the property of double orthogonality shown here to be possessed by these functions, and to present some numerical results illustrating them. It will be assumed throughout that the parameter c is real and positive, although many of the results are more general. A secondary purpose is to emphasize the underlying unity of the properties associated with the spheroidal functions of arbitrary real order $\alpha > -1$. All properties of the spheroidal wave functions and of the periodic Mathieu functions arise quite naturally from those of the general functions $\psi_{\alpha n}(c, \eta)$. An extensive and completely general treatment for all solutions of (1) for arbitrary complex values of α, b, and c has been developed by Meixner and Schäfke [3].

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2. Elementary Properties

The following properties of the spheroidal functions $\psi_{\alpha n}(c, \eta)$ are evident directly from either the differential equation (1) and its boundary condition of finiteness at $\eta = \pm 1$ or the equivalent integral equation (2).

(a) The functions $\psi_{\alpha n}(c, \eta)$ are entire functions of η. This follows from the fact that the parameter $b_{\alpha n}$ was chosen in such a way that one of the two solutions of differential equation (1) remains finite at the singular points $\eta = \pm 1$. With no singularities except for the one at infinity that solution $\psi_{\alpha n}(c, \eta)$ must be entire.

(b) As $c \to 0$ the functions $\psi_{\alpha n}(c, \eta)$ become proportional to the Gegenbauer functions $T_n^{\alpha}(\eta)$, and the eigenvalues $b_{\alpha n}(c)$ approach

$$b_{\alpha n}(c) \to n(n+2\alpha+1). \tag{3}$$

This follows from the fact that the differential equation (1) approaches the Gegenbauer differential equation [4, p. 783]

$$(1-\eta^2)T_n^{\alpha\prime\prime}(\eta) - 2(\alpha+1)\eta T_n^{\alpha\prime}(\eta) + n(n+2\alpha+1)T_n^{\alpha}(\eta) = 0. \tag{4}$$

The Gegenbauer functions for $\alpha = m = 0, 1, 2, 3, \ldots$ are related to the associated Legendre functions $P_{m+n}^{m}(\eta)$ by

$$T_n^{m}(\eta) = \frac{P_{m+n}^{m}(\eta)}{(1-\eta^2)^{m/2}}, \tag{5}$$

and for $\alpha = -\frac{1}{2}$ and $+\frac{1}{2}$ they are related to the Chebyshev polynomials of the first and second kinds, respectively, by

$$nT_n^{-\frac{1}{2}}(\eta) = \sqrt{\frac{2}{\pi}} \cos\left[n \cos^{-1}\eta\right], \tag{6}$$

$$T_n^{\frac{1}{2}}(\eta) = \sqrt{\frac{2}{\pi}} \frac{\sin\left[(n+1)\cos^{-1}\eta\right]}{\sqrt{1-\eta^2}}. \tag{7}$$

(c) For $c \to 0$ and $\eta \to \infty$ such that $c\eta$ remains finite the function $\psi_{\alpha n}(c, \eta)$ becomes proportional to $J_{n+\alpha+\frac{1}{2}}(c\eta)/(c\eta)^{\alpha+\frac{1}{2}}$. This follows from the integral representation (2) in which $\psi_{\alpha n}(c, t)$ in the interval $-1 \leq t \leq 1$ becomes proportional to $T_n^{\alpha}(\eta)$, from the fact [1(20)] that

$$\int_{-1}^{1} e^{ic\eta t}(1-t^2)^{\alpha} T_n^{\alpha}(t)\,dt = \frac{\sqrt{2\pi}\, i^n \Gamma(n+2\alpha+1)}{n!} \frac{J_{n+\alpha+\frac{1}{2}}(c\eta)}{(c\eta)^{\alpha+\frac{1}{2}}}, \tag{8}$$

for $\alpha > -1$.

(d) The functions $\psi_{\alpha n}(c, \eta)$ are real for real η, have exactly n zeros in the interval $(-1, 1)$, and are even or odd functions of η according as n is even or odd. This is implied (not proved) by the limiting case $c \to 0$ in which the limiting functions, the Gegenbauer functions of (4), are known from Sturm-Liouville theory to be real, to have exactly n zeros in $(-1, 1)$, and to be even or odd functions of η according as n is even or odd.

(e) The function $\psi_{\alpha n}(c, \eta)$ cannot have a zero at $\eta = \pm 1$. This follows from the differential equation (1) and the entirety of $\psi_{\alpha n}(c, \eta)$. For if it were to have a zero then, from (1), the first derivative also must be zero there. And after l successive differentiations of (1) every $l+1$th derivative

must also be zero there, which would require that the function $\psi_{\alpha n}(c, \eta)$ be identically zero everywhere in η.

(f) The eigenvalues $b_{\alpha n}(c)$ of (1) are real, positive, and ordered such that $b_{\alpha 0} < b_{\alpha 1} < b_{\alpha 2} < \ldots$, provided that they are distinct. This is implied (again not proved) by the limiting case $c \to 0$, being given in the limit by (3).

(g) The functions $\psi_{\alpha n}(c, \eta)$ behave asymptotically with real $\eta \to +\infty$ as

$$\psi_{\alpha n}(c, \eta) = \frac{i^n 2^{\alpha+1}\Gamma(\alpha+1)\psi_{\alpha n}(c, 1)}{\nu_{\alpha n}(c)} \frac{\cos\left(c\eta - \frac{n+\alpha+1}{2}\pi\right)}{(c\eta)^{\alpha+1}}\left[1+O\left(\frac{1}{\eta}\right)\right]. \tag{9}$$

This follows from the known asymptotic behavior [5] of integrals of the form (2).

(h) The spheriodal wave functions $S_{m,\,m+n}(c, \eta)$ and the periodic Mathieu functions $Se_n(c, \eta)$ and $So_{n+1}(c, \eta)$ are special cases related to $\psi_{\alpha n}(c, \eta)$ as follows[2]:

$$\psi_{\alpha n}(c, \eta) = \begin{cases} \dfrac{S_{m,\,m+n}(c, \eta)}{(1-\eta^2)^{m/2}}, & \alpha = m = 0, 1, 2, 3, \ldots \\ Se_n(c, \eta), & \alpha = -\dfrac{1}{2} \\ \dfrac{So_{n+1}(c, \eta)}{(1-\eta^2)^{1/2}}, & \alpha = +\dfrac{1}{2}. \end{cases} \tag{10}$$

This is evident from the integral equation for the spheroidal wave functions [6(4)] and for the periodic Mathieu functions [6(11, 12)].

3. Construction of the Functions

On the interval $(-1, 1)$ the Gegenbauer functions $T_k^\alpha(\eta)$ form a complete set with respect to all functions that are square-integrable with weight factor $(1-\eta^2)^\alpha$, and are orthogonal with weight factor $(1-\eta^2)^\alpha$ [1(6)]:

$$\int_{-1}^{1} (1-\eta^2)^\alpha T_k^\alpha(\eta) T_l^\alpha(\eta) d\eta = \frac{2\Gamma(k+2\alpha+1)}{(2k+2\alpha+1)k!}\delta_{kl}. \tag{11}$$

They are even or odd functions according as k is even or odd. Hence the spheroidal functions $\psi_{\alpha n}(c, \eta)$ can be expanded in Gegenbauer functions on $(-1, 1)$ [1(8)],

$$\psi_{\alpha n}(c, \eta) = \sum_{k=0,\,1}^{\infty}{}' d_k(c|\alpha n) T_k^\alpha(\eta), \tag{12}$$

where the prime denotes summation over only even or odd values of k according as n is even or odd. The function $\psi_{\alpha n}(c, \eta)$ becomes proportional to $T_n^\alpha(\eta)$ as $c \to 0$, hence

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[2] The connection between the notation of Meixner and Schäfke [3] and that used here is as follows:

$$b_{\alpha n}(c) = \lambda^\alpha_{\alpha+n}(c^2) - \alpha(\alpha+1) + c^2, \qquad \psi_{\alpha n}(c, \eta) \propto (1-\eta^2)^{-\alpha/2} ps^{-\alpha}_{\alpha+n}(\eta; c^2);$$

$$b_{-\frac{1}{2},\,n}(c) = \lambda_n\left(\frac{c^2}{4}\right) + \frac{c^2}{2}, \qquad \psi_{-\frac{1}{2},\,n}(c, \eta) \propto ce_n\left(\cos^{-1}\eta; \frac{c^2}{4}\right);$$

$$b_{\frac{1}{2},\,n}(c) = \lambda_{-n-1}\left(\frac{c^2}{4}\right) - 1 + \frac{c^2}{2}, \qquad \psi_{\frac{1}{2},\,n}(c, \eta) \propto (1-\eta^2)^{-\frac{1}{2}} se_{n+1}\left(\cos^{-1}\eta; \frac{c^2}{4}\right).$$

$$d_k(c|\alpha n) \to 0,\ k \neq n. \tag{13}$$

Expansion (12) converges rapidly on the interval $(-1, 1)$, but beyond that interval convergence becomes slower and slower with increasing η. An alternative expansion that represents the functions uniformly on $(-\infty, \infty)$, by an argument used later for (40), can be obtained from (12) in the integral equation (2) by integrating term by term and making use of (8),

$$\psi_{\alpha n}(c, \eta) = \frac{i^n\sqrt{2\pi}}{\nu_{\alpha n}(c)} \sum_{k=0,1}^{\infty}{}' a_k(c|\alpha n)\, \frac{J_{k+\alpha+\frac{1}{2}}(c\eta)}{(c\eta)^{\alpha+\frac{1}{2}}}, \tag{14}$$

where the expansion coefficients a_k are related to the d_k by

$$a_k(c|\alpha n) = i^{k-n}\, \frac{\Gamma(k+2\alpha+1)}{k!}\ d_k(c|\alpha n). \tag{15}$$

The coefficients a_k are all real because the d_k are real (the factor i^{k-n} is always real since $k-n$ is always even). As $c \to 0$,

$$a_k(c|\alpha n) \to 0,\ k \neq n, \tag{16}$$

by (13). For the special cases of $\alpha = m = 0, 1, 2, 3, \ldots$ the coefficients a_k will agree with those tabulated by Stratton et al. [7] for the prolate spheroidal wave functions provided that the normalization is chosen to be such that

$$\sum_{k=0,1}^{\infty}{}' i^{k-n} a_k(c|\alpha n) = 1. \tag{17}$$

With the right hand side of (17) replaced by $\sqrt{\pi/2}$ the coefficients a_k for the special cases of $\alpha = -\frac{1}{2}$ and $+\frac{1}{2}$ are related to Blanch's [8] Mathieu function coefficients $De_k^{(n)}$ and $Do_{k+1}^{(n+1)}$, respectively, by

$$\begin{aligned} a_k(c|-\tfrac{1}{2}, n) &= i^{k-n}\sqrt{\frac{\pi}{2}}\, De_k^{(n)}(c^2), \\ a_k(c|+\tfrac{1}{2}, n) &= i^{k-n}\sqrt{\frac{\pi}{2}}\,(k+1)\, Do_{k+1}^{(n+1)}(c^2). \end{aligned} \tag{18}$$

In addition to the increasingly poor convergence of (12) outside of $(-1, 1)$ it requires special consideration for the case of $\alpha = -\frac{1}{2}$ because $T_k^{-\frac{1}{2}}(\eta)$ is infinite for $k = 0$. Hence the expansion (14) will be used almost exclusively hereafter.

A recursion relationship [1(9)] for determining the a_k's in (14) and the eigenvalues $b_{\alpha n}(c)$ is obtained from the differential equation (1),

$$A_{k+2}a_{k+2} + (b_{\alpha n} - B_k)a_k + C_{k-2}a_{k-2} = 0, \tag{19}$$

where

$$A_k=\frac{k(k-1)}{(2k+2\alpha-1)(2k+2\alpha+1)}\,c^2,$$

$$B_k=k(k+2\alpha+1)+\frac{2k^2+2k(2\alpha+1)+2\alpha-1}{(2k+2\alpha-1)(2k+2\alpha+3)}\,c^2,$$

$$C_k=\frac{(k+2\alpha+1)(k+2\alpha+2)}{(2k+2\alpha+1)(2k+2\alpha+3)}\,c^2.$$

It can be rewritten as a continued fraction for the ratio of two succeeding values of a_k in two different ways, one in terms of increasing subscript,

$$\frac{a_k}{a_{k-2}}=-\frac{C_{k-2}}{b_{\alpha n}-B_k+A_{k+2}\dfrac{a_{k+2}}{a_k}}, \tag{20}$$

and the other in terms of decreasing subscript,

$$\frac{a_{k-2}}{a_k}=-\frac{A_k}{b_{\alpha n}-B_{k-2}+C_{k-4}\dfrac{a_{k-4}}{a_{k-2}}}. \tag{21}$$

Each must be the reciprocal of the other, which leads to an infinite set of eigenvalues for $b_{\alpha n}$. The best method now available for computing these eigenvalues, and simultaneously the expansion coefficients a_k, is an iterative procedure that was first used by Bouwkamp [9(48)] for the case $\alpha=0$ and later invented independently by Blanch [10] for the Mathieu functions. All of the coefficients a_k in (19) must vanish for negative k, according to a result of Gegenbauer's [11, Sec 16.13] in which he showed that any analytic function can be represented within its domain of analyticity by a series of the form (14). And for $k=0$ or 1 the A_k, B_k, C_k in (19) become indeterminate at $\alpha=\pm\frac{1}{2}$. Therefore (19) should be supplemented by

$$\begin{gathered} a_k=0 \qquad \text{for} \qquad k<0, \\ B_0=\frac{c^2}{2\alpha+3},\ B_1=2\alpha+2+\frac{3c^2}{2\alpha+5},\ C_0=\frac{2\alpha+2}{2\alpha+3}\,c^2, \end{gathered} \tag{22}$$

the latter of which resolves the indeterminacy. For the special case of the Mathieu functions ($\alpha=\pm\frac{1}{2}$) as computed by Chu and Stratton [2(100)], where $A_k=C_k=\frac{c^2}{4}$ and $B_k=k^2+\frac{c^2}{2}$, the two computational procedures are quite different although the numerical results have been found to be exactly the same. The advantage of the new procedure here is that all orders $\alpha>-1$ are treated exactly alike.

Once the eigenvalues are known the ratio $\frac{a_k}{a_{k-2}}$ for all k is also known, either from (20) or (21). The a_k's can then be computed from these ratios in terms of a_0 or a_1 for the even and odd spheroidal functions, respectively. The particular value chosen for a_0 or a_1 determines the scale factor by which the $\psi_{\alpha n}(c,\eta)$ functions are normalized. But no common agreement exists among authors on the choice of normalization factors. For this reason *all formulas in this paper will be shown with arbitrary normalization factors* that can be chosen at will. For those numerical results shown whose values depend upon a choice of scale factor the $\psi_{\alpha n}(c,\eta)$ functions will be normalized to unity over the interval $(-\infty,\infty)$.

The eigenvalues $\nu_{\alpha n}(c)$ can be computed by equating the two expressions (12) and (14), or their derivatives, at $\eta=0$. As η approaches zero the Bessel functions in (14) cause all terms to vanish except for $k=0$. Hence for even $n=2r$,

$$\nu_{\alpha,2r}(c)=\frac{(-1)^r\sqrt{\pi}\,a_0(c|\alpha,2r)}{2^\alpha\Gamma(\alpha+\frac{3}{2})\psi_{\alpha,2r}(c,0)}. \tag{23}$$

For odd $n=2r+1$ the $\psi_{\alpha n}(c,\eta)$ vanish at $\eta=0$, so (14) must be differentiated with respect to η before letting η approach zero. All terms vanish except for $k=1$, hence

$$\nu_{\alpha,2r+1}(c)=ic\,\frac{(-1)^r\sqrt{\pi}\,a_1(c|\alpha,2r+1)}{2^{\alpha+1}\Gamma(\alpha+\frac{5}{2})\psi'_{\alpha,2r+1}(c,0)}. \tag{24}$$

The $\psi_{\alpha n}(c,\eta)$ functions and their derivatives at $\eta=0$ are found from (12) to be

$$\psi_{\alpha,2r}(c,0)=\frac{(-1)^r}{2^\alpha\sqrt{\pi}}\sum_{l=0}^{\infty}\frac{\Gamma(l+\frac{1}{2})}{\Gamma(l+\alpha+1)}\,a_{2l}(c|\alpha,2r), \tag{25}$$

$$\psi'_{\alpha,2r+1}(c,0)=\frac{(-1)^r}{2^{\alpha-1}\sqrt{\pi}}\sum_{l=0}^{\infty}\frac{\Gamma(l+\frac{3}{2})}{\Gamma(l+\alpha+1)}\,a_{2l+1}(c|\alpha,2r+1). \tag{26}$$

Also,

$$\nu_{\alpha n}(c)=i^n|\nu_{\alpha n}(c)|. \tag{27}$$

4. Double Orthogonality

An important property that will be established here is that the spheroidal functions $\psi_{\alpha n}(c,\eta)$ for any order $\alpha>-1$ are orthogonal with weight factor $|1-\eta^2|^\alpha$ on two different intervals $(-1,1)$ and $(-\infty,\infty)$ simultaneously. This double orthogonality property was first recognized for the case of the prolate spheroidal wave functions of order zero by Slepian and Pollak [12]. A later investigation by the author [6] revealed that this property was possessed by all of the prolate spheroidal wave functions of order $m=0,1,2,3,\ldots$ and by the periodic Mathieu functions.

The differential equation for the functions $S_{\alpha,\alpha+n}(c,\eta)$ defined by

$$\psi_{\alpha n}(c,\eta)=\frac{S_{\alpha,\alpha+n}(c,\eta)}{(1-\eta^2)^{\alpha/2}} \tag{28}$$

is found from (1) to be of the singular Sturm-Liouville form

$$[(1-\eta^2)S'_{\alpha,\alpha+n}(c,\eta)]'+\left[b_{\alpha n}+\alpha(\alpha+1)-c^2\eta^2-\frac{\alpha^2}{1-\eta^2}\right]S_{\alpha,\alpha+n}(c,\eta)=0. \tag{29}$$

Multiplication by $S^*_{\alpha,\alpha+p}(c,\eta)$ and subtraction of the product equation obtained when $S_{\alpha,\alpha+n}(c,\eta)$ and $S^*_{\alpha,\alpha+p}(c,\eta)$ are interchanged gives

$$\frac{d}{d\eta}\left[(1-\eta^2)(S'_{\alpha,\alpha+n}S^*_{\alpha,\alpha+p}-S_{\alpha,\alpha+n}S^{*\prime}_{\alpha,\alpha+p})\right]=(b_{\alpha p}-b_{\alpha n})S_{\alpha,\alpha+n}S^*_{\alpha,\alpha+p}. \tag{30}$$

Integrating with respect to η over a real interval and changing from $S_{\alpha,\alpha+n}(c,\eta)$ back to $\psi_{\alpha n}(c,\eta)$,

$$(1-\eta^2)|1-\eta^2|^\alpha[\psi'_{\alpha n}(c,\eta)\psi_{\alpha p}(c,\eta)-\psi_{\alpha n}(c,\eta)\psi'_{\alpha p}(c,\eta)]\Big|_{\eta=\eta_1}^{\eta_2}$$
$$=(b_{\alpha p}-b_{\alpha n})\int_{\eta_1}^{\eta_2}\psi_{\alpha n}(c,\eta)\psi_{\alpha p}(c,\eta)|1-\eta^2|^\alpha d\eta. \tag{31}$$

The functions $\psi_{\alpha n}(c,\eta)$ will be orthogonal over any real interval (η_1,η_2) for which the left hand

side of (31) is zero, provided that the b_{an} are distinct. Distinctness of the b_{an} can usually be determined numerically.

(a) *Orthogonality on* $(-1, 1)$

When $\eta_1=-1$ and $\eta_2=1$ the left hand side of (31) vanishes for all $\alpha>-1$ because $(1-\eta^2)|1-\eta^2|^\alpha$ vanishes at both end points and $\psi_{an}(c, \eta)$ and its derivative are finite. Hence, if the b_{an} are distinct,

$$\int_{-1}^{1} \psi_{an}(c, \eta)\psi_{ap}(c, \eta)(1-\eta^2)^\alpha d\eta=0, \qquad n\neq p; \tag{32}$$

i.e., the $\psi_{an}(c, \eta)$ functions are orthogonal on $(-1, 1)$ with weight factor $(1-\eta^2)^\alpha$.

(b) *Orthogonality on* $(-\infty, \infty)$

From the asymptotic behavior (9) of $\psi_{an}(c, \eta)$ a simple calculation shows that

$$(1-\eta^2)|1-\eta^2|^\alpha[\psi'_{an}(c, \eta)\psi_{ap}(c, \eta)-\psi_{an}(c, \eta)\psi'_{ap}(c, \eta)]$$
$$=\frac{\psi_{an}(c, 1)\psi_{ap}(c, 1)4^{\alpha+1}\Gamma^2(\alpha+1)}{|\nu_{an}(c)||\nu_{ap}(c)|c^{2\alpha+1}}\left[\sin\frac{p-n}{2}\pi+O\left(\frac{1}{\eta}\right)\right], \tag{33}$$

as $\eta\to+\infty$, η real. When $p-n$ is even the first term is zero for all η, leaving

$$(1-\eta^2)|1-\eta^2|^\alpha[\psi'_{an}(c, \eta)\psi_{ap}(c, \eta)-\psi_{an}(c, \eta)\psi'_{ap}(c, \eta)]=O(1/\eta).$$

The left hand side of (31) then vanishes for $\eta_1=1$ and $\eta_2=\infty$, hence

$$\int_{1}^{\infty} \psi_{an}(c, \eta)\psi_{ap}(c, \eta)|1-\eta^2|^\alpha d\eta=0, \qquad p-n \text{ even and } \neq 0. \tag{34}$$

Similarly on the interval $(-\infty, -1)$, because of symmetry of the functions. Combining the three intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$,

$$\int_{-\infty}^{\infty} \psi_{an}(c, \eta)\psi_{ap}(c, \eta)|1-\eta^2|^\alpha d\eta=0, \qquad p\neq n, \tag{35}$$

including $p-n$ odd because $\psi_{an}(c, \eta)\psi_{ap}(c, \eta)$ is then an odd function on $(-\infty, \infty)$. Thus the functions $\psi_{an}(c, \eta)$ are orthogonal with weight factor $|1-\eta^2|^\alpha$ on the interval $(-\infty, \infty)$, provided only that the b_{an} are distinct.

5. Completeness

(a) *Completeness on* $(-1, 1)$

The functions $S_{\alpha,\alpha+n}(c, \eta)$ are complete on the interval $(-1, 1)$ with respect to all square-integrable functions. This is a property [13] of all Sturm-Liouville systems of the form

$$(pu')'+(\lambda w-q)u=0, \quad puu'|_a=puu'|_b=0, \tag{36}$$

and hence applies to the differential equation (29) and the boundary condition that $S_{\alpha,\alpha+n}(c, \eta)$ must remain finite at the singular points $\eta = \pm 1$. Completeness of the $S_{\alpha,\alpha+n}(c, \eta)$ functions implies completeness of the $\psi_{\alpha n}(c, \eta)$ functions.

(b) *Completeness on* $(-\infty, \infty)$

The fact that the functions $\psi_{\alpha n}(c, \eta)$ are complete on $(-1, 1)$ with respect to the class of functions that are square-integrable with weight factor $(1-\eta^2)^\alpha$ will now be used to show that they are also complete on $(-\infty, \infty)$ with respect to all functions of the form

$$F_\alpha(c\eta) = \int_{-1}^{1} e^{ic\eta t}(1-t^2)^\alpha f(t)\,dt \tag{37}$$

for $\alpha > -1$. The only restriction on $f(t)$ is that it be square-integrable on $(-1, 1)$ with weight factor $(1-t^2)^\alpha$. It can be expanded in a series that converges in the mean with weight factor $(1-t^2)^\alpha$,

$$f(t) = \sum_{n=0}^{\infty} \frac{b_n(c)}{\nu_{\alpha n}(c)} \psi_{\alpha n}(c, t), \qquad -1 < t < 1, \tag{38}$$

where

$$b_n(c) = \frac{\nu_{\alpha n}(c)}{\Lambda_{\alpha n}(c)} \int_{-1}^{1} f(t)\psi_{\alpha n}(c, t)(1-t^2)^\alpha dt. \tag{39}$$

Hence for *all* real values of η the absolute value

$$\left|F_\alpha(c\eta) - \sum_{n=0}^{\infty} b_n\psi_{\alpha n}(c, \eta)\right| = \left|\int_{-1}^{1} e^{ic\eta t}(1-t^2)^\alpha \left[f(t) - \sum_{n=0}^{\infty} \frac{b_n}{\nu_{\alpha n}} \psi_{\alpha n}(c, t)\right] dt\right|$$

$$\leq \int_{-1}^{1} (1-t^2)^\alpha \left|f(t) - \sum_{n=0}^{\infty} \frac{b_n}{\nu_{\alpha n}} \psi_{\alpha n}(c, t)\right| dt \tag{40}$$

is exactly zero. It is concluded, therefore, that

$$F_\alpha(c\eta) = \sum_{n=0}^{\infty} b_n(c)\psi_{\alpha n}(c, \eta), \qquad -\infty < \eta < \infty, \tag{41}$$

where the series converges to $F_\alpha(c\eta)$ not just in the mean but uniformly on $(-\infty, \infty)$. Thus the functions $\psi_{\alpha n}(c, \eta)$ are complete on $(-\infty, \infty)$ with respect to all functions $F_\alpha(c\eta)$ of the form (37).

6. Normalization Factors

The double orthogonality property of the spheroidal functions results in two different sets of normalization factors, one on $(-1, 1)$ and another on $(-\infty, \infty)$. Both will be developed here.

(a) *Normalization factors on* $(-1, 1)$.

The orthonormality relation on $(-1, 1)$ will be designated by

$$\int_{-1}^{1} \psi_{\alpha n}(c, \eta)\psi_{\alpha p}(c, \eta)(1-\eta^2)^\alpha d\eta = \Lambda_{\alpha n}(c)\delta_{np}, \tag{42}$$

where $\Lambda_{\alpha n}(c)$ denotes the normalization factor for the spheroidal function of order α and index n. These normalization factors can be computed directly from the integral (42) for each $p = n$ by

expanding $\psi_{\alpha n}(c, \eta)$ in Gegenbauer functions as in (12) and making use of their orthogonality relation (11), obtaining

$$\Lambda_{\alpha n}(c) = \sum_{k=0,1}^{\infty}{}' \frac{k!}{(k+\alpha+\frac{1}{2})\Gamma(k+2\alpha+1)} a_k^2(c|\alpha n), \tag{43}$$

where the a_k's are the expansion coefficients (15).

(b) *Normalization factors on* $(-\infty, \infty)$.

Each of the normalization factors on $(-\infty, \infty)$ can be derived in terms of the corresponding normalization factor $\Lambda_{\alpha n}(c)$ on $(-1, 1)$ by making use of the double orthogonality property already established. The orthogonality integral (35) can be written

$$\int_{-\infty}^{\infty} \psi_{\alpha n}(c, \eta)\psi_{\alpha p}(c, \eta)|1-\eta^2|^{\alpha} d\eta$$

$$= \int_{-1}^{1} \left[\frac{1}{\nu_{\alpha p}(c)} \int_{-\infty}^{\infty} e^{ic\eta t}|1-\eta^2|^{\alpha}\psi_{\alpha n}(c, \eta) d\eta \right] (1-t^2)^{\alpha}\psi_{\alpha p}(c, t) dt, \tag{44}$$

by replacing $\psi_{\alpha p}(c, \eta)$ by its integral representation (2) and interchanging the order of integration. From the orthogonality property (35) on $(-\infty, \infty)$ it is clear that this must be zero for all $p \neq n$. And from the other orthogonality property (32) on $(-1, 1)$ it follows that the bracketed expression in (44) must be orthogonal to every $\psi_{\alpha p}(c, t)$ except $\psi_{\alpha n}(c, t)$. Then, from completeness on $(-1, 1)$, it is concluded that the bracketed expression for $p = n$ must be proportional to $\psi_{\alpha n}(c, t)$ everywhere on $(-1, 1)$:

$$\frac{1}{\nu_{\alpha n}(c)} \int_{-\infty}^{\infty} e^{ic\eta t}|1-\eta^2|^{\alpha}\psi_{\alpha n}(c, \eta) d\eta \equiv \gamma_{\alpha n}(c)\psi_{\alpha n}(c, t), \qquad -1 < t < 1, \tag{45}$$

where $\gamma_{\alpha n}(c)$ is the proportionality constant.[3] Thus the orthogonality relation on $(-\infty, \infty)$ is

$$\int_{-\infty}^{\infty} \psi_{\alpha n}(c, \eta)\psi_{\alpha p}(c, \eta)|1-\eta^2|^{\alpha} d\eta = \gamma_{\alpha n}(c)\Lambda_{\alpha n}(c)\delta_{np}; \tag{46}$$

that is, each normalization factor on $(-\infty, \infty)$ is proportional to the corresponding normalization factor on $(-1, 1)$, the proportionality factor being the real positive number $\gamma_{\alpha n}(c)$ defined by (45).

The new numbers $\gamma_{\alpha n}(c)$ representing the ratio of the two normalization factors are characteristic of the functions $\psi_{\alpha n}(c, \eta)$, much like the eigenvalues $b_{\alpha n}(c)$ and $\nu_{\alpha n}(c)$, even though their defining equation (45) is not an integral equation in any ordinary sense. However they are by far the most difficult of the three to evaluate numerically for arbitrary $\alpha > -1$. In a few special cases, notably $\alpha = -\frac{1}{2}, 0, \frac{1}{2}$, and 1, it becomes practical to evaluate them by evaluating the normalization integral (46). Such a method was actually used to obtain the first published values for the case of the prolate spheroidal wave functions of order zero [12, p. 59] (they were not recognized there as being a new kind of characteristic numbers because for $\alpha = 0$ they are not, being related simply to the eigenvalues of integral equation (2)) and the Mathieu functions [6 (60, 61)]. But for arbitrary α the only practical method to evaluate them appears to be from their definition (45),

$$\gamma_{\alpha n}(c) = \frac{\int_{-\infty}^{\infty} e^{ic\eta t}|1-\eta^2|^{\alpha}\psi_{\alpha n}(c, \eta) d\eta}{\nu_{\alpha n}(c)\psi_{\alpha n}(c, t)}, \qquad -1 < t < 1, \tag{47}$$

[3] An early draft of the present paper was communicated privately to J. Meixner, who became sufficiently concerned about the lack of convergence of the integral in (45) for $\alpha \geq 1$ that he looked into the possibility of an alternative derivation. From some unpublished notes that he kindly sent me after this paper was submitted for publication it appears that he has succeeded in deriving the relationship (45) by an entirely different approach. He has given the integral from $-\infty$ to -1 and from 1 to ∞ a meaning by assuming that t has a negative or positive imaginary part, respectively, and then approaching the limit of real t, thereby avoiding the difficult problem of proving summability.

at some convenient value of t in the interval $(-1, 1)$. In particular, at $t = 0$ this reduces to

$$\gamma_{\alpha,2r}(c) - 1 = \frac{2\int_1^\infty (\eta^2 - 1)^\alpha \psi_{\alpha,2r}(c, \eta)\, d\eta}{\nu_{\alpha,2r}(c)\psi_{\alpha,2r}(c, 0)} \tag{48}$$

for even $n = 2r$. For odd $n = 2r + 1$ the ratio (47) becomes indeterminate at $t = 0$ because $\psi_{\alpha,2r+1}(c, \eta)$ is an odd function of η. It can be made determinate by applying L'Hospital's rule at $t = 0$,

$$\gamma_{\alpha,2r+1}(c) - 1 = \frac{i2c\int_1^\infty \eta(\eta^2 - 1)^\alpha \psi_{\alpha,2r+1}(c, \eta)\, d\eta}{\nu_{\alpha,2r+1}(c)\psi'_{\alpha,2r+1}(c, 0)} \tag{49}$$

for $n = 2r + 1$. The problem of evaluating $\gamma_{\alpha n}(c)$ thus reduces to evaluation of the integrals in (48) and (49).

The above method for determining $\gamma_{\alpha n}(c)$ was introduced originally by the author [6(42)] for the special case of the spheroidal wave functions of integral order $\alpha = m$. It has been confirmed and generalized recently in an elegant and quite different way by Meixner [14], who showed that for integral $\alpha = m$ an expression equivalent to (45) holds for the most general solutions of the differential equation (1) for arbitrary complex values of c, of b, and of the characteristic exponent ν in place of the integer n.

The integral in (48) does not converge at infinity for $\alpha \geqslant 1$, nor does the one in (49) for $\alpha \geqslant 0$, but they both appear to be Cesàro summable $(C, m + 1)$ for all $-1 < \alpha \leqq m$ because of the sinusoidal asymptotic behavior (9) of $\psi_{\alpha n}(c, \eta)$. They can be evaluated by expanding $\psi_{\alpha n}(c, \eta)$ in Bessel functions by (14) and integrating term by term,

$$\gamma_{\alpha,2r}(c) - 1 = \frac{2\sqrt{2\pi}}{|\nu^2_{\alpha,2r}(c)\psi_{\alpha,2r}(c, 0)|} \sum_{p=0}^{\infty} a_{2p}(c|\alpha, 2r) I_{\alpha,2p}(c), \tag{50}$$

$$\gamma_{\alpha,2r+1}(c) - 1 = \frac{2\sqrt{2\pi}}{|\nu^2_{\alpha,2r+1}(c)\psi'_{\alpha,2r+1}(c, 0)|} \sum_{p=0}^{\infty} a_{2p+1}(c|\alpha, 2r + 1) K_{\alpha,2p}(c), \tag{51}$$

where the integrals $I_{\alpha k}(c)$ and $K_{\alpha k}(c)$ are defined by

$$I_{\alpha k}(c) = \int_1^\infty (\eta^2 - 1)^\alpha \frac{J_{k+\alpha+\frac{1}{2}}(c\eta)}{(c\eta)^{\alpha+\frac{1}{2}}}\, d\eta, \tag{52}$$

$$K_{\alpha k}(c) = \int_1^\infty (\eta^2 - 1)^\alpha \frac{J_{k+2+\alpha-\frac{1}{2}}(c\eta)}{(c\eta)^{\alpha-\frac{1}{2}}}\, d\eta. \tag{53}$$

For $k = 0$ the integral $I_{\alpha 0}(c)$ in the range $-1 < \alpha < 0$ can be evaluated by replacing the Bessel function in (52) by its representation [11, p. 48(2)] as an integral on (0, 1) and interchanging the order of integration,

$$I_{\alpha 0}(c) = \frac{2^{-\alpha+\frac{1}{2}}}{\Gamma(\alpha + 1)\Gamma(\frac{1}{2})} \int_0^1 (1 - t^2)^\alpha \left[\int_1^\infty (\eta^2 - 1)^\alpha \cos c\eta t\, d\eta\right] dt. \tag{54}$$

The inner integral on $(1, \infty)$ is an integral representation for the Neumann function [11, p. 170(4)], hence

$$I_{\alpha 0}(c) = -\int_0^1 (1-t^2)^\alpha \frac{Y_{-(\alpha+\frac{1}{2})}(ct)}{(ct)^{\alpha+\frac{1}{2}}}\,dt = -\frac{1}{\cos\alpha\pi}\left[\sin\alpha\pi\int_0^1 (1-t^2)^\alpha \frac{J_{-(\alpha+\frac{1}{2})}(ct)}{(ct)^{\alpha+\frac{1}{2}}}\,dt + \int_0^1 (1-t^2)^\alpha \frac{J_{\alpha+\frac{1}{2}}(ct)}{(ct)^{\alpha+\frac{1}{2}}}\,dt\right], \tag{55}$$

the latter expression coming from the definition of the Neumann function [11, p. 64(1)]. Expanding the Bessel functions in power series about the origin and integrating term by term gives

$$I_{\alpha 0}(c) = \frac{\Gamma(\alpha+1)}{2^{\alpha+\frac{1}{2}}\cos\alpha\pi}\sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{l!}\left[\frac{\sin\alpha\pi\Gamma(l-\alpha)}{l!\Gamma(l-\alpha+\frac{1}{2})}\left(\frac{c}{2}\right)^{-2\alpha-1} + \frac{\Gamma(l+\frac{1}{2})}{\Gamma^2(l+\alpha+\frac{3}{2})}\right]\left(\frac{c}{2}\right)^{2l}. \tag{56}$$

The integrals $I_{\alpha k}(c)$ and $K_{\alpha k}(c)$ for all k can now be obtained from $I_{\alpha 0}(c)$ by recursion in k. By use of the recursion relationships among Bessel functions [11, p. 45(1,2)] the following recursion relationships among $I_{\alpha k}(c)$ and $K_{\alpha k}(c)$ can be established, valid for all $\alpha > -1$:

$$K_{\alpha,k-2}(c) + K_{\alpha k}(c) = (2k+2\alpha+1)I_{\alpha k}(c), \tag{57}$$

$$K_{\alpha,k-2}(c) - K_{\alpha k}(c) = (2\alpha+1)I_{\alpha k}(c) + 2cI'_{\alpha k}(c), \tag{58}$$

$$K_{\alpha k}(c) = kI_{\alpha k}(c) - cI'_{\alpha k}(c), \tag{59}$$

$$(k+2\alpha+3)I_{\alpha,k+2}(c) + cI'_{\alpha,k+2}(c) = kI_{\alpha k}(c) - cI'_{\alpha k}(c). \tag{60}$$

The last relation (60) is a differential equation for $I_{\alpha,k+2}(c)$ in terms of $I_{\alpha k}(c)$. Its solution is

$$I_{\alpha,k+2}(c) = \frac{2k+2\alpha+3}{c^{k+2\alpha+3}}\int_0^c t^{k+2\alpha+2}I_{\alpha k}(t)\,dt - I_{\alpha k}(c). \tag{61}$$

Repeated application of (61) to $I_{\alpha 0}(c)$ in (56) suggests the appropriate form of the power series expansion for $I_{\alpha,2p}(c)$, which is then easily shown by induction to hold for all nonnegative integers p. Its jth derivative, obtained by differentiating term by term with respect to c, is

$$I^{(j)}_{\alpha,2p}(c) = \frac{\Gamma(\alpha+1)}{2^{j+\alpha+\frac{1}{2}}\cos\alpha\pi}\sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{l!}\left[\frac{\sin\alpha\pi\Gamma(l-\alpha)(-l+\alpha+\frac{1}{2})_p\,(2l-2\alpha-j)_j}{(l+p)!\Gamma(l-\alpha+\frac{1}{2})}\left(\frac{c}{2}\right)^{-2\alpha-1} + \frac{\Gamma(l+\frac{1}{2})(-l)_p(2l+1-j)_j}{\Gamma(l+\alpha+\frac{3}{2})\Gamma(l+\alpha+\frac{3}{2}+p)}\right]\left(\frac{c}{2}\right)^{2l-j}, \tag{62}$$

where $(z)_p = z(z+1)(z+2)\ \ldots\ (z+p-1)$, $(z)_0 = 1$, is Pochhammer's symbol. $I_{\alpha,2p}(c)$ and $K_{\alpha,2p}(c)$ for $-1 < \alpha < 0$ can thus be computed by (62) and (59), respectively.

To obtain $I_{\alpha,2p}(c)$ and $K_{\alpha,2p}(c)$ for orders beyond the region of integrability $-1 < \alpha < 0$ of (55) the following recursion relations are valid for all $\alpha > 0$,

$$I_{\alpha k}(c) = \left(\frac{k}{c^2} - \frac{1}{2k+2\alpha+1}\right)I_{\alpha-1,k}(c) - \frac{1}{2k+2\alpha+1}I_{\alpha-1,k+2}(c) - \frac{1}{c}I'_{\alpha-1,k}(c), \tag{63}$$

$$K_{\alpha k}(c) = \frac{k(k+2)}{c^2}I_{\alpha-1,k}(c) - I_{\alpha-1,k+2}(c) - \frac{2k+1}{c}I'_{\alpha-1,k}(c) + I''_{\alpha-1,k}(c), \tag{64}$$

the first being obtained from the definition (52) and the Bessel function recursion relations and the second from (59). It has been found by direct substitution that the series (62) satisfies the recursion

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relation (63). From this it is concluded that (62) is valid for *all* $\alpha > -1$, even though the integral (55) from which it was derived contains a nonintegrable singularity at $t=0$ for all nonintegral values of α greater than zero. As an indication of the rapidity of convergence of (62) and of (50) and (51), and hence of the practicality of the formulas developed here, the series (62) always was found to converge to 16 significant figures in less than 23 terms for the computations shown later, and the series (50) and (51) to converge to 16 figures in less than 19 terms.

The series (62) becomes indeterminate when α is an integer or a half-integer. The indeterminancy at any of the integers $\alpha=m$ can be resolved by taking the limit as α approaches m. For $\alpha=0$, for example, the limiting value $I_{00}(c)$ is

$$I_{00}(c)=\frac{1}{c}\sqrt{\frac{\pi}{2}}-\sqrt{\frac{2}{\pi}}\sum_{l=0}^{\infty}(-1)^l\ \frac{c^{2l}}{(2l+1)(2l+1)!}, \tag{65}$$

in agreement with the power series expansion of

$$\frac{1}{c}\sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2}-Si\ c\right)$$

obtained by direct evaluation of the integral (52). At $\alpha=-\frac{1}{2}$ the indeterminancy again can be resolved by taking the limit, obtaining

$$I_{-\frac{1}{2},0}(c)=\frac{1}{2\sqrt{\pi}}\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(l+\frac{1}{2})}{(l!)^3}\left[3\psi(l+1)-\psi(l+\tfrac{1}{2})-2\ln\frac{c}{2}\right]\left(\frac{c}{2}\right)^{2l} \tag{66}$$

in terms of the Digamma function $\psi(z)$, which is identical to the power series expansion [11, p. 150 (1)] of

$$-\frac{\pi}{2}\,J_0\left(\frac{c}{2}\right)Y_0\left(\frac{c}{2}\right)$$

obtained by direct evaluation of the integral (52). But in the special cases of integral and half-integral orders, and these cases only, there are more direct means available for evaluating $\gamma_{\alpha n}(c)$ than by the series (50) and (51). For all integral orders the numbers $\gamma_{\alpha n}(c)$ are found to be expressible quite simply in terms of the spheroidal eigenvalues $b_{\alpha n}(c)$ of the differential equation and $\nu_{\alpha n}(c)$ of the integral equation [6(23)]. For the half-integral orders $\alpha=\pm\frac{1}{2}$ the numbers $\gamma_{\alpha n}(c)$ have been found [6(31)] in terms of Blanch's joining factors $f_{e,n}(c^2)$ and $f_{o,n+1}\ (c^2)$ to be simply

$$\gamma_{\alpha n}(c)=\begin{cases}1+f_{e,n}(c^2), & \alpha=-\frac{1}{2}\\ 1+f_{o,n+1}(c^2), & \alpha=+\frac{1}{2}.\end{cases} \tag{67}$$

From Blanch's expansion of $f_{e,n}(c^2)$ and $f_{o,n+1}(c^2)$ in terms of Bessel functions it can be shown that (67) is identical to that obtained from (50) and (51). The integrals $I_{\alpha k}(c)$ and $K_{\alpha k}(c)$ can be expressed in closed form for half-integral values of α, starting from $I_{-\frac{1}{2},2p}(c)$ [15(6.552.6)]

$$I_{-\frac{1}{2},2p}\ (c)=-\frac{\pi}{2}\,J_p\left(\frac{c}{2}\right)Y_p\left(\frac{c}{2}\right), \tag{68}$$

from which $K_{-\frac{1}{2},2p}(c)$ is obtained by recursion relation (59),

$$K_{-\frac{1}{2},2p}\ (c)=-\frac{\pi c}{4}\left[J_p\left(\frac{c}{2}\right)Y_{p+1}\left(\frac{c}{2}\right)+J_{p+1}\left(\frac{c}{2}\right)Y_p\left(\frac{c}{2}\right)\right], \tag{69}$$

and from which $I_{\frac{1}{2},2p}(c)$ and $K_{\frac{1}{2},2p}(c)$ are obtained by recursion relations (63) and (59), respectively,

$$I_{\frac{1}{2},2p}(c)=\frac{\pi}{4(2p+1)}\left[J_p\left(\frac{c}{2}\right)Y_p\left(\frac{c}{2}\right)+J_{p+1}\left(\frac{c}{2}\right)Y_{p+1}\left(\frac{c}{2}\right)\right.$$
$$\left.-\frac{2p+1}{c}\left(J_p\left(\frac{c}{2}\right)Y_{p+1}\left(\frac{c}{2}\right)+J_{p+1}\left(\frac{c}{2}\right)Y_p\left(\frac{c}{2}\right)\right)\right], \tag{70}$$

$$K_{\frac{1}{2},2p}(c)=\frac{\pi}{4}\left[J_p\left(\frac{c}{2}\right)Y_p\left(\frac{c}{2}\right)+J_{p+1}\left(\frac{c}{2}\right)Y_{p+1}\left(\frac{c}{2}\right)\right.$$
$$\left.-\frac{2p+2}{c}\left(J_p\left(\frac{c}{2}\right)Y_{p+1}\left(\frac{c}{2}\right)+J_{p+1}\left(\frac{c}{2}\right)Y_p\left(\frac{c}{2}\right)\right)\right]. \tag{71}$$

7. An Extremal Property

An important property of the prolate spheroidal wave functions of order zero was established by Landau and Pollak [16, Th. 1], namely, that the first $N+1$ of them are the $N+1$ linearly independent functions of the form $F_0(c\eta)$ that are most concentrated in the interval $(-1, 1)$. By the term "concentration" of a function of the form $F_\alpha(c\eta)$ defined by (37) will be meant the value of the ratio $R_\alpha(c)$,

$$R_\alpha(c)=\frac{\int_{-1}^{1}|F_\alpha(c\eta)|^2(1-\eta^2)^\alpha d\eta}{\int_{-\infty}^{\infty}|F_\alpha(c\eta)|^2|1-\eta^2|^\alpha d\eta}. \tag{72}$$

It will now be shown, by quite a different and much more simple argument, that for *any* order $\alpha>-1$ the first $N+1$ of the spheroidal functions $\psi_{\alpha n}(c,\eta)$ are the $N+1$ linearly independent functions of the form $F_\alpha(c\eta)$ that are most concentrated in the interval $(-1, 1)$.

After expanding $F_\alpha(c\eta)$ in spheroidal functions as in (41) and making use of the two orthogonality relations (42) and (46) the concentration ratio (72) becomes

$$R_\alpha=\frac{\sum_{n=0}^{\infty}b_n b_n^*\Lambda_{\alpha n}}{\sum_{n=0}^{\infty}b_n b_n^*\gamma_{\alpha n}\Lambda_{\alpha n}}. \tag{73}$$

It is determined entirely by the expansion coefficients b_n of the function $F_\alpha(c\eta)$. Hence a necessary condition for it to attain its largest value is that it be an extremum with respect to the real and the imaginary parts of each and every one of the b_n's; i.e., since b_n and b_n^* are linearly related to the real and imaginary parts of b_n,

$$\frac{\partial R_\alpha}{\partial b_p}=\frac{\partial R_\alpha}{\partial b_p^*}=0, \qquad p=0, 1, 2, 3, \ldots. \tag{74}$$

Rewriting (73) in the form

$$\sum_{n=0}^{\infty}b_n b_n^*(R_\alpha\gamma_{\alpha n}-1)\Lambda_{\alpha n}=0, \tag{75}$$

then differentiating through with respect to b_p or b_p^* and imposing condition (74),

$$b_p^*(R_\alpha\gamma_{\alpha p}-1)\Lambda_{\alpha p}=b_p(R_\alpha\gamma_{\alpha p}-1)\Lambda_{\alpha p}=0, \qquad p=0, 1, 2, 3, \ldots. \tag{76}$$

The normalization factors $\Lambda_{\alpha p}(c)$ are never zero, hence either

$$b_p = b_p^* = 0, \tag{77}$$

or

$$R_\alpha \gamma_{\alpha p} - 1 = 0. \tag{78}$$

The second condition (78) can be true for at most only one value of p, in which case condition (74) for all other values of p requires that (77) be true. It is clear, then, that an extremum of $R_\alpha(c)$ can occur when, and only when, $F_\alpha(c\eta)$ is proportional to one of the spheroidal functions $\psi_{\alpha n}(c, \eta)$. The value of $R_\alpha(c)$ at each of these extrema is, from (78),

$$R_{\alpha_{\text{ext}}}(c) = \frac{1}{\gamma_{\alpha n}(c)}, \qquad n = 0, 1, 2, 3, \ldots . \tag{79}$$

The largest possible value of $R_\alpha(c)$ is that for which $\gamma_{\alpha n}(c)$ is least, which is $\gamma_{\alpha 0}(c)$. Hence the spheroidal function for $n = 0$ is the single most concentrated function of the form $F_\alpha(c\eta)$.

By the term "linear independence" of a set of functions is meant that they are mutually orthogonal. Thus to find the second most concentrated function of the form $F_\alpha(c\eta)$ that is linearly independent of $\psi_{\alpha 0}(c, \eta)$ on $(-\infty, \infty)$ means to find the most concentrated function of the form $F_\alpha(c\eta)$ that is orthogonal to $\psi_{\alpha 0}(c, \eta)$ on $(-\infty, \infty)$. From the orthogonality of the functions $\psi_{\alpha n}(c, \eta)$ on $(-\infty, \infty)$ and from their completeness with respect to functions of the form $F_\alpha(c\eta)$ it follows that all functions of the form $F_\alpha(c\eta)$ that are linearly independent of $\psi_{\alpha 0}(c, \eta)$ can be expanded in the set of $\psi_{\alpha n}(c, \eta)$ functions that remain after $\psi_{\alpha 0}(c, \eta)$ has been removed; i.e., after setting $b_0 = 0$ in (41). Repeating the maximization process above leads again to conditions (77) and (78) but restricted now to $p = 1, 2, 3 \ldots$. The largest possible value for $R_\alpha(c)$ is again that for which $\gamma_{\alpha n}(c)$ is least, which now is $\gamma_{\alpha 1}(c)$. Hence the spheroidal functions for $n = 0$ and 1 are the most concentrated *pair* of linearly independent functions of the form $F_\alpha(c\eta)$. By this process of systematically depleting the set of functions $\psi_{\alpha n}(c, \eta)$ it is concluded that the first $N+1$ of them are indeed the $N+1$ linearly independent functions of the form $F_\alpha(c\eta)$ that are most concentrated in the interval $(-1, 1)$.

8. Spheroidal Identities

All of the identities to be derived here will be obtained from the integral equation (2) and the following property of completeness of the functions $\psi_{\alpha n}(c, \eta)$ on $(-1, 1)$:

$$\delta(t - t') = (1 - t^2)^\alpha \sum_{n=0}^{\infty} \frac{\psi_{\alpha n}(c, t)\psi_{\alpha n}(c, t')}{\Lambda_{\alpha n}(c)}, \tag{80}$$

for $-1 < t < 1$ and $-1 < t' < 1$, obtained by expanding the delta function $\delta(t - t')$ in the orthonormal functions $\psi_{\alpha n}(c, t')/\sqrt{\Lambda_{\alpha n}(c)}$ with expansion coefficients $(1 - t^2)^\alpha \psi_{\alpha n}(c, t)/\sqrt{\Lambda_{\alpha n}(c)}$. Although such an expansion is open to question on grounds that the delta function is not square-integrable it does appear to be valid as the limiting case of some square-integrable function that approaches the delta function.

The principal identity from which the others will be derived is obtained by multiplying the integral equation (2) for argument η by its complex conjugate for some other argument y, then dividing by the normalization factor $\Lambda_{\alpha n}(c)$, and finally summing over all n,

$$\sum_{n=0}^{\infty} \frac{|\nu_{\alpha n}(c)|^2}{\Lambda_{\alpha n}(c)} \psi_{\alpha n}(c, \eta)\psi_{\alpha n}(c, y) = \int_{-1}^{1}\int_{-1}^{1} e^{ic(\eta t - yt')}(1 - t'^2)^\alpha \left[(1 - t^2)^\alpha \sum_{n=0}^{\infty} \frac{\psi_{\alpha n}(c, t)\psi_{\alpha n}(c, t')}{\Lambda_{\alpha n}(c)} \right] dt\,dt'. \tag{81}$$

The bracketed expression is $\delta(t-t')$, by (80), which reduces the right hand side to an integral representation of a Bessel function [11, p. 48(4)]. Hence one obtains the identity

$$\sum_{n=0}^{\infty} \frac{|\nu_{\alpha n}(c)|^2}{\Lambda_{\alpha n}(c)} \psi_{\alpha n}(c,\eta)\psi_{\alpha n}(c,y) = 2^{\alpha+\frac{1}{2}}\,\Gamma(\alpha+1)\Gamma(\tfrac{1}{2})\,\frac{J_{\alpha+\frac{1}{2}}(c[\eta-y])}{(c[\eta-y])^{\alpha+\frac{1}{2}}} \tag{82}$$

for $-\infty<\eta<\infty$, $-\infty<y<\infty$.

An important special case of (82) occurs when $y=\eta$, giving

$$\sum_{n=0}^{\infty} \frac{|\nu_{\alpha n}(c)|^2}{\Lambda_{\alpha n}(c)} \psi_{\alpha n}^2(c,\eta) = \frac{\Gamma(\alpha+1)\Gamma(\frac{1}{2})}{\Gamma(\alpha+\frac{3}{2})} \tag{83}$$

for $-\infty<\eta<\infty$. At $\eta=0$ the product $\nu_{\alpha n}(c)\psi_{\alpha n}(c,0)$ for odd n is zero and for even n is proportional to a_0, by (23), hence

$$\sum_{r=0}^{\infty} \frac{a_0^2(c|\alpha,2r)}{\Lambda_{\alpha,2r}(c)} = \tfrac{1}{2}\Gamma(2\alpha+2). \tag{84}$$

This identity provides a means for an overall numerical check on computed values of the normalization factors $\Lambda_{\alpha n}(c)$ alone for even n. To obtain the corresponding identity for odd n construct a new identity like (83),

$$\sum_{n=0}^{\infty} \frac{|\nu_{\alpha n}(c)|^2}{\Lambda_{\alpha n}(c)} \psi'^2_{\alpha n}(c,\eta) = c^2\,\frac{\Gamma(\alpha+1)\Gamma(\frac{3}{2})}{\Gamma(\alpha+\frac{5}{2})} \tag{85}$$

for $-\infty<\eta<\infty$. At $\eta=0$ this reduces, by (24), to

$$\sum_{r=0}^{\infty} \frac{a_1^2(c|\alpha,2r+1)}{\Lambda_{\alpha,2r+1}(c)} = (\alpha+\tfrac{3}{2})\Gamma(2\alpha+2), \tag{86}$$

which provides a means for checking numerical values of the $\Lambda_{\alpha n}(c)$'s alone for odd n.

An identity for the eigenvalues $\nu_{\alpha n}(c)$ alone can be obtained by multiplying (83) by $(1-\eta^2)^{\alpha}$ and integrating over $(-1,1)$,

$$\sum_{n=0}^{\infty} |\nu_{\alpha n}(c)|^2 = \left(\frac{\Gamma(\alpha+1)\Gamma(\frac{1}{2})}{\Gamma(\alpha+\frac{3}{2})}\right)^2. \tag{87}$$

An identity for checking the new numbers $\gamma_{\alpha n}(c)$ for even n can be obtained from (82) by letting $y=0$, multiplying by $(\eta^2-1)^{\alpha}$, and integrating over $(1,\infty)$,

$$\sum_{r=0}^{\infty} (-1)^r \frac{|\nu_{\alpha,2r}(c)|^3\psi^2_{\alpha,2r}(c,0)}{\Lambda_{\alpha,2r}(c)}\,[\gamma_{\alpha,2r}(c)-1] = 2^{\alpha+\frac{3}{2}}\,\Gamma(\alpha+1)\Gamma(\tfrac{1}{2})\,I_{\alpha 0}(c), \tag{88}$$

where $I_{\alpha 0}(c)$ is as defined by (52). For odd n differentiate (82) before letting $y=0$ and then multiply by $\eta(\eta^2-1)^{\alpha}$ and integrate over $(1,\infty)$,

$$\sum_{r=0}^{\infty} (-1)^{r+1} \frac{|\nu_{\alpha,2r+1}(c)|^3\psi'^2_{\alpha,2r+1}(c,0)}{\Lambda_{\alpha,2r+1}(c)}\,[\gamma_{\alpha,2r+1}(c)-1] = 2^{\alpha+\frac{3}{2}}\,\Gamma(\alpha+1)\Gamma(\tfrac{1}{2})c^2 I'_{\alpha 0}(c). \tag{89}$$

Another identity can be constructed by multiplying integral equation (2) by $\psi_{an}(c, y)/\sqrt{\Lambda_{an}(c)}$ and summing,

$$\sum_{n=0}^{\infty} \frac{\nu_{an}(c)}{\Lambda_{an}(c)} \psi_{an}(c, \eta)\psi_{an}(c, y) = e^{ic\eta y}, \tag{90}$$

for $-\infty < \eta < \infty$, $-1 < y < 1$. For $y=0$ this becomes, from (23) and (24),

$$\sum_{r=0}^{\infty} (-1)^r \frac{a_0(c|\alpha, 2r)}{\Lambda_{\alpha, 2r}(c)} \psi_{\alpha, 2r}(c, \eta) = \frac{2^{\alpha}\Gamma(\alpha+\frac{1}{2})}{\Gamma(\frac{1}{2})}, \tag{91}$$

$$\sum_{r=0}^{\infty} (-1)^r \frac{a_1(c|\alpha, 2r+1)}{\Lambda_{\alpha, 2r+1}(c)} \psi_{\alpha, 2r+1}(c, \eta) = \frac{2^{\alpha+1}\Gamma(\alpha+\frac{3}{2})}{\Gamma(\frac{1}{2})} \eta, \tag{92}$$

for $-\infty < \eta < \infty$.

In the limiting case where $c \to 0$ and η is such that $c\eta = \zeta$,

$$\psi_{an}(c, \eta) \to \frac{n!}{\Gamma(n+2\alpha+1)} a_n T_n^{\alpha}(\eta), \tag{93}$$

$$\Lambda_{an}(c) \to \frac{n!}{(n+\alpha+\frac{1}{2})\Gamma(n+2\alpha+1)} a_n^2, \tag{94}$$

$$|\nu_{an}(c)|\psi_{an}(c, \eta) \to \sqrt{2\pi}\, a_n \frac{J_{n+\alpha+\frac{1}{2}}(\zeta)}{\zeta^{\alpha+\frac{1}{2}}}. \tag{95}$$

Then (82) for $c \to 0$ with $c\eta = \zeta$ and $cy = z$ reduces to

$$\sum_{n=0}^{\infty} \frac{(n+\alpha+\frac{1}{2})\Gamma(n+2\alpha+1)}{n!} \frac{J_{n+\alpha+\frac{1}{2}}(\zeta)}{\zeta^{\alpha+\frac{1}{2}}} \frac{J_{n+\alpha+\frac{1}{2}}(z)}{z^{\alpha+\frac{1}{2}}} = \frac{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)\Gamma(\frac{1}{2})}{2\pi} \frac{J_{\alpha+\frac{1}{2}}(\zeta-z)}{(\zeta-z)^{\alpha+\frac{1}{2}}}, \tag{96}$$

which is one of Gegenbauer's addition theorems [11, p. 363(2)], and (90) for $c \to 0$ with $c\eta = \zeta$ reduces to

$$\sqrt{2\pi} \sum_{n=0}^{\infty} i^n (n+\alpha+\tfrac{1}{2}) \frac{J_{n+\alpha+\frac{1}{2}}(\zeta)}{\zeta^{\alpha+\frac{1}{2}}} T_n^{\alpha}(y) = e^{i\zeta y}, \tag{97}$$

which is another [11, p. 368(2)].

9. A Special Case

For the special case in which $c = \frac{q\pi}{2}$, $\alpha = 1$, and $n = q-1$, with $q = 1, 2, 3, \ldots$, an exact closed-form representation can be obtained for the spheroidal function and each of its associated numbers. Its importance lies in the fact that it provides an independent means for an exact numerical check.

The spheroidal function sought can be obtained from the differential equation (1) and the condition of finiteness at $\eta = \pm 1$ by letting

$$\psi_{an}(c, \eta) = \frac{u_{an}(c, \eta)}{1-\eta^2}, \tag{98}$$

which leads to

$$(1-\eta^2)u''_{\alpha n}-2(\alpha-1)\eta u'_{\alpha n}+\left(b_{\alpha n}+2-c^2\eta^2-\frac{4(\alpha-1)\eta^2}{1-\eta^2}\right)u_{\alpha n}=0. \tag{99}$$

When $\alpha=1$ this reduces to

$$(1-\eta^2)u''_{1n}+(b_{1n}+2-c^2\eta^2)u_{1n}=0. \tag{100}$$

For $c=\frac{q\pi}{2}$ and assuming for the moment that

$$b_{1,q-1}\left(\frac{q\pi}{2}\right)=\left(\frac{q\pi}{2}\right)^2-2, \tag{101}$$

(100) becomes, for $n=q-1$,

$$u''_{1,q-1}+\left(\frac{q\pi}{2}\right)^2 u_{1,q-1}=0 \tag{102}$$

whose solution is

$$u_{1,q-1}\left(\frac{q\pi}{2},\eta\right)=k_q \begin{matrix}\cos\\ \sin\end{matrix}\frac{q\pi}{2}\eta, \tag{103}$$

hence

$$\psi_{1,q-1}\left(\frac{q\pi}{2},\eta\right)=k_q\frac{\begin{matrix}\cos\\ \sin\end{matrix}\frac{q\pi}{2}\eta}{1-\eta^2},\qquad q\begin{matrix}\text{odd}\\ \text{even}\end{matrix}, \tag{104}$$

where k_q is a normalizing constant. Expression (104) satisfies the differential equation (1) and is finite at $\eta=\pm 1$, so it is clearly a spheroidal function. Furthermore it has exactly $q-1$ zeros in the interval $(-1, 1)$, so the index of the spheroidal function must be $q-1$. Thus the validity of (101) is established. The spheroidal function (104) and its eigenvalue (101) were obtained earlier by Flammer [17].

The spheroidal function (104) satisfies also the integral equation (2), which can be verified by direct calculation of the integral,

$$\int_{-1}^{1} e^{i\frac{q\pi}{2}\eta}\,(1-t^2)\frac{\begin{matrix}\cos\\ \sin\end{matrix}\frac{q\pi}{2}t}{1-t^2}\,dt=i^{q-1}\,\frac{4}{q\pi}\,\frac{\begin{matrix}\cos\\ \sin\end{matrix}\frac{q\pi}{2}\eta}{1-\eta^2}. \tag{105}$$

This is the integral equation in question, from which it is evident that its eigenvalue is

$$\nu_{1,q-1}\left(\frac{q\pi}{2}\right)=i^{q-1}\frac{4}{q\pi}. \tag{106}$$

The normalization factor on $(-1, 1)$ in terms of the constant k_q is found by direct calculation of the integral (42) to be

$$\Lambda_{1,q-1}\left(\frac{q\pi}{2}\right)=\frac{k_q^2}{2}\,\mathrm{Cin}\,2q\pi. \tag{107}$$

And by direct calculation of the integral (46) one finds

$$\gamma_{1,q-1}\left(\frac{q\pi}{2}\right)=2. \tag{108}$$

The above results provide an interesting indication of the way in which the integral in the definition (45) of $\gamma_{\alpha n}(c)$ can behave outside of the open interval $-1<t<1$. For q odd, for example, the left-hand side of (45) can be evaluated in closed form,

$$(-1)^{\frac{q-1}{2}}\frac{q\pi}{4}\int_{-\infty}^{\infty}e^{i\frac{q\pi}{2}\eta t}\,|1-\eta^2|\,\frac{\cos\frac{q\pi}{2}\eta}{1-\eta^2}\,d\eta$$

$$=2\,\frac{\cos\frac{q\pi}{2}t}{1-t^2}-(-1)^{\frac{q-1}{2}}\frac{\pi}{2}[\delta(t-1)+\delta(t+1)], \tag{109}$$

which agrees exactly with the right-hand side of (45) within the open interval $-1<t<1$. It does not agree at the endpoints $t=\pm 1$, however, as evidenced by the delta function singularities that appear there.

10. Numerical Results

The formulas developed above have been used to compute and check the spheroidal functions and their associated numbers for values of c up to 5π. Representative numerical results for $c=6$ are shown below, all of which were computed on the Triangle Universities IBM 360/75 computer by Guy A. Myers at the North Carolina State University.

The first nine of the spheroidal functions $\psi_{\alpha n}(c,\eta)$ are shown in figure 1 for $\alpha=-\frac{1}{2}, 0, \frac{1}{2}, 1, 2,$ and 3. They were normalized on the interval $(-\infty, \infty)$ by choosing $\Lambda_{\alpha n}(c)$ to be $1/\gamma_{\alpha n}(c)$. The corresponding functions for $\alpha=0$ with $c=0.5$, 1, 2, and 4 were shown earlier by Slepian and Pollak [12]. The characteristic numbers $b_{\alpha n}(c)$, $\nu_{\alpha n}(c)$, and $\gamma_{\alpha n}(c)$ as a function of α are shown in figures 2, 3, and 4, respectively.

Numerical results for a few representative cases are shown in tables 1 and 2. Table 1 shows computed values of $b_{\alpha n}(c)$, $|\nu_{\alpha n}(c)|$, and $\psi_{\alpha n}(c,0)$ for even n and $\psi'_{\alpha n}(c,0)$ for odd n, for $\alpha=-\frac{1}{2}$, 0, and $+\frac{3}{4}$ at $c=6$, $n=0$ to 13. Table 2 shows computed values for the reciprocal of $\gamma_{\alpha n}(c)$ for $\alpha=-\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1,$ and $\frac{5}{4}$, at $c=6$, $n=0$ to 13. The reciprocal of $\gamma_{\alpha n}(c)$, rather than $\gamma_{\alpha n}(c)$ itself, is shown in order to illustrate the relative concentration (79) of the functions.

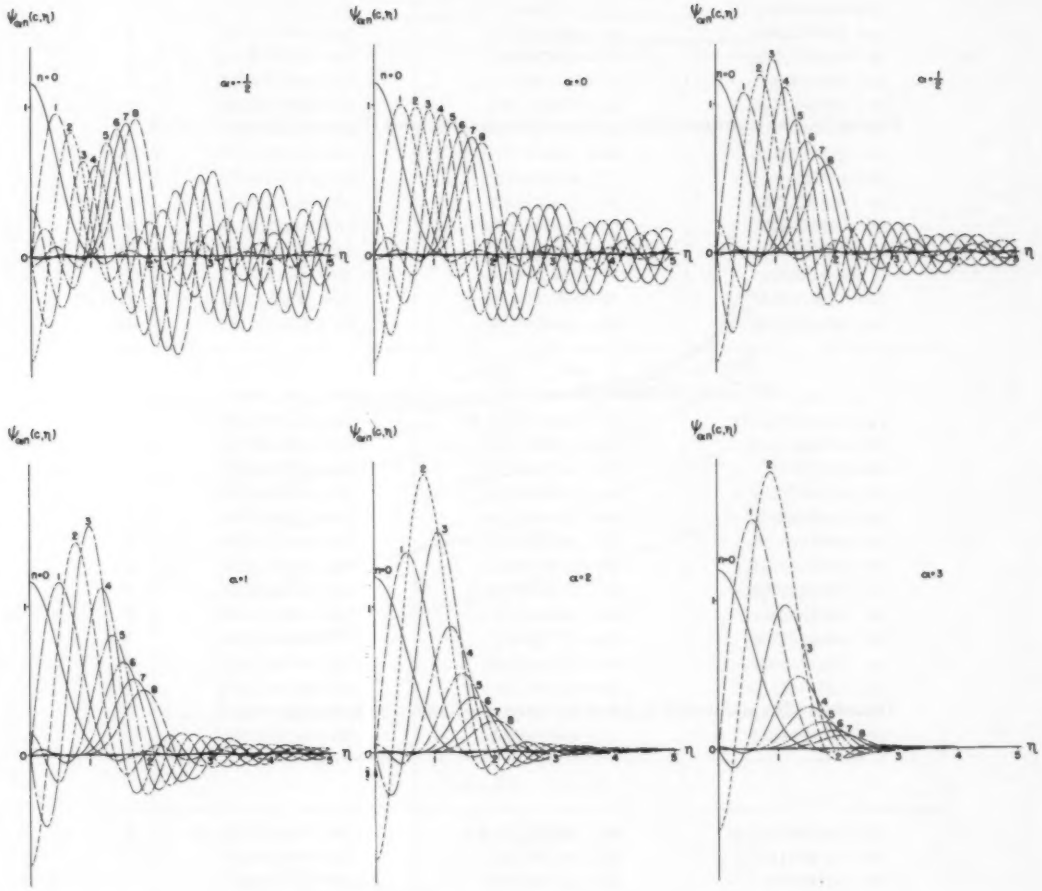

FIGURE 1. *The spheroidal functions for* $\alpha = -1/2, 0, 1/2, 1, 2, 3$, $c = 6$, $n = 0$ to 8, *with* $\Lambda_{\alpha n}(c) = 1/\gamma_{\alpha n}(c)$.

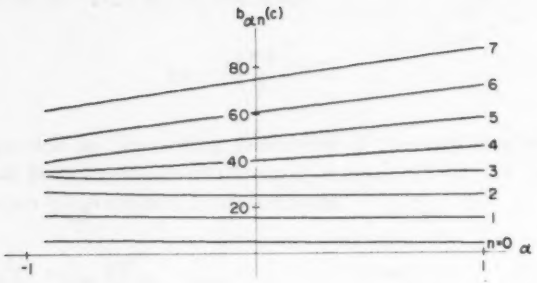

FIGURE 2. *The eigenvalues* $b_{\alpha n}(c)$ *of the differential equation* (1) *for* $-0.92 \leq \alpha \leq 1$, $c = 6$.

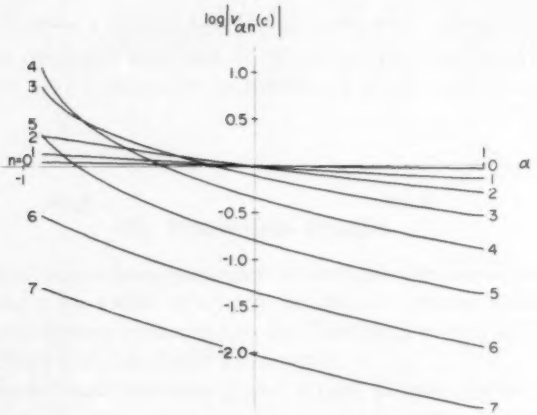

FIGURE 3. *The eigenvalues* $\nu_{\alpha n}(c)$ *of the integral equation* (2) *for* $-0.92 \leq \alpha \leq 1$, $c = 6$.

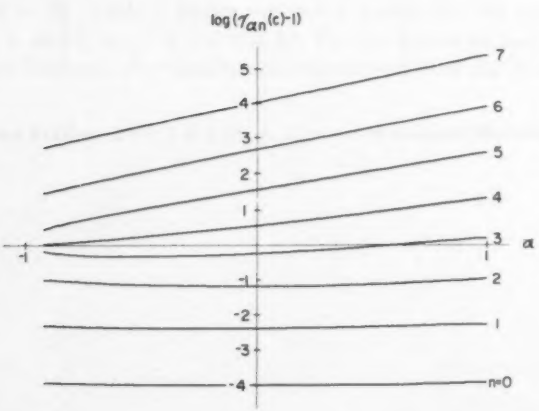

FIGURE 4. *The new numbers* $\gamma_{\alpha n}(c)$ *defined by* (45) *for* $-0.92 \leq \alpha \leq 1$, $c = 6$.

TABLE 1

n	$b_{an}(c)$	$\lvert\nu_{an}(c)\rvert$	$\psi_{an}(c, 0)$ $\psi'_{an}(c, 0)$
		$\alpha=-1/2, \quad c=6$	
0	0.5737585781 + 01	0.1074447786 + 01	0.1155255760 + 01
1	.1661329843 + 02	.1212324184 + 01	.3686243582 + 01
2	.2598284316 + 02	.1469587054 + 01	−.6927129388 + 00
3	.3290367966 + 02	.1770806514 + 01	−.2666895609 + 01
4	.3816092638 + 02	.1311549557 + 01	.3176164384 + 00
5	.4509186607 + 02	.4656092304 + 00	.1273298762 + 01
6	.5521934384 + 02	.1117504031 + 00	−.6410649522 − 01
7	.6785668990 + 02	.2310300898 − 01	−.1308328538 + 00
8	.8264753135 + 02	.4243576673 − 02	.3284620306 − 02
9	.9950842599 + 02	.6981239327 − 03	.5920679880 − 02
10	.1184102166 + 03	.1037825612 − 03	−.9512187852 − 04
11	.1393381242 + 03	.1405973769 − 04	−.1623325826 − 03
12	.1622835823 + 03	.1748761994 − 05	.1808459022 − 05
13	.1872412953 + 03	.2010009631 − 06	.2990162450 − 05
		$\alpha=0, \quad c=6$	
0	0.5208269159 + 01	0.1023276503 + 01	0.1153430212 + 01
1	.1600044275 + 02	.1021309607 + 01	.3662488818 + 01
2	.2535647864 + 02	.9922435547 + 00	−.7045705538 + 00
3	.3320419949 + 02	.8229938914 + 00	−.2977793219 + 01
4	.4072019426 + 02	.4659781026 + 00	.2864309440 + 00
5	.4977371212 + 02	.1693510360 + 00	.7958373899 + 00
6	.6118075689 + 02	.4524678973 − 01	−.3152431639 − 01
7	.7485286652 + 02	.9966212672 − 02	−.6222320400 − 01
8	.9065115936 + 02	.1897100693 − 02	.1388242306 − 02
9	.1085154453 + 03	.3192404741 − 03	.2472548565 − 02
10	.1284188799 + 03	.4820488557 − 04	−.3606307265 − 04
11	.1503474434 + 03	.6605196410 − 05	−.6113574747 − 04
12	.1742930029 + 03	.8286721621 − 06	.6275570287 − 06
13	.2002505133 + 03	.9588946460 − 07	.1033409530 − 05
		$\alpha=+3/4, \quad c=6$	
0	0.4524868470 + 01	0.9514693801 + 00	0.1157683567 + 01
1	.1554416968 + 02	.7960998515 + 00	.3741186029 + 01
2	.2556039130 + 02	.6000021716 + 00	−.7319049126 + 00
3	.3502025958 + 02	.3687113331 + 00	−.2915597899 + 01
4	.4493678220 + 02	.1690739803 + 00	.1800330864 + 00
5	.5631678092 + 02	.5762489345 − 01	.3736554493 + 00
6	.6963816767 + 02	.1546055800 − 01	−.1125847285 − 01
7	.8500027646 + 02	.3450023959 − 02	−.2071854851 − 01
8	.1024006606 + 03	.6633536963 − 03	.3879838010 − 03
9	.1218275965 + 03	.1124131264 − 03	.6700408720 − 03
10	.1432726639 + 03	.1705749626 − 04	−.8491880927 − 05
11	.1667303958 + 03	.2345366168 − 05	−.1414240391 − 04
12	.1921972059 + 03	.2949700757 − 06	.1289490027 − 06
13	.2196706817 + 03	.3419275479 − 07	.2099163320 − 06

TABLE 2

Reciprocal of $\gamma_{\alpha n}(c)$ for $c=6$

n	$\alpha=-3/4$	$\alpha=-1/2$	$\alpha=-1/4$
0	0.9998903116+00	0.9998968961+00	0.9999006633+00
1	.9953591572+00	.9957735784+00	.9959984111+00
2	.9251077487+00	.9361280337+00	.9403541758+00
3	.6617807432+00	.6867880908+00	.6773573623+00
4	.4329978267+00	.3559391049+00	.2783892137+00
5	.1631458857+00	.8787799051−01	.4890053549−01
6	.1812019930−01	.8375304321−02	.4009596286−02
7	.1085823478−02	.4742740153−03	.2109141235−03
8	.4704479809−04	.1948721509−04	.8155744837−05
9	.1566525909−05	.6167222470−06	.2444286528−06
10	.4138876101−07	.1554061438−07	.5862729444−08
11	.8891403564−09	.3195449002−09	.1152410426−09
12	.1584025115−10	.5466463632−11	.1891529381−11
13	.2378498905−12	.7904581458−13	.2632527573−13
n	$\alpha=0$	$\alpha=+1/4$	$\alpha=+1/2$
0	0.9999018826+00	0.9999006439+00	0.9998968665+00
1	.9960616432+00	.9959753979+00	.9957367356+00
2	.9401733901+00	.9363969465+00	.9290353072+00
3	.6467919491+00	.5993323516+00	.5374082010+00
4	.2073492168+00	.1474458621+00	.1005292486+00
5	.2738716624−01	.1530259540−01	.8498117614−02
6	.1955000733−02	.9617772918−03	.4747110329−03
7	.9484876556−04	.4292801980−04	.1948802000−04
8	.3436783286−05	.1454316917−05	.6167231119−06
9	.9732115988−07	.3886092897−07	.1554061496−07
10	.2218980545−08	.8416064934−09	.3195449004−09
11	.4166226283−10	.1508525403−10	.5466463632−11
12	.6557478591−12	.2276034041−12	.7904581458−13
13	.8780377122−14	.2931302844−14	.9790506399−15
n	$\alpha=+3/4$	$\alpha=+1$	$\alpha=+5/4$
0	0.9998902878+00	0.9998804332+00	0.9998665602+00
1	.9953272353+00	.9947106203+00	.9938277694+00
2	.9175535778+00	.9009322432+00	.8776555430+00
3	.4640663909+00	.3839718982+00	.3032540474+00
4	.6605749034−01	.4205022091−01	.2605099768−01
5	.4681647613−02	.2555876781−02	.1381915965−02
6	.2342324766−03	.1152638318−03	.5647610537−04
7	.8852297954−05	.4016362284−05	.1817713005−05
8	.2616675768−06	.1109412441−06	.4695584821−07
9	.6217024284−08	.2485740919−08	.9925660313−09
10	.1213549245−09	.4606583810−10	.1746778699−10
11	.1981169523−11	.7177204946−12	.2597757239−12
12	.2745434078−13	.9531862327−14	.3306789087−14
13	.3270100000−15	.1091853480−15	.3643090846−16

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Automatic Computing Methods for Special Functions

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Some of the pitfalls of automatic computation are described as well as their possible avoidance. A detailed description is given of the methods for computing the error and complementary error function. The implementation of the method is given in a double precision ASA FORTRAN program. The test ("driver") program and test results are also given.

Key words: Approximations; computer programs; computing pitfalls; continued fractions; error function; recurrence relations; series.

1. Introduction

AMS55[1][1]—the Handbook of Mathematical Functions—in addition to defining functions in terms of their integrals, power series, asymptotic expansions, etc., also supplies tables and approximating polynomials as well as methods of computing values both within and outside the tabular range. With the present trend to large scale computer systems, questions frequently arise concerning methods of computing the tabular values themselves. In many, if not most cases, the method used for computing the table is certainly not the best method for computing discrete values. While many specialized methods and programs exist [3, 6, 7, 8], as a supplement to the Handbook and to satisfy general needs, a systematic effort is being made to supply methods of computing special functions, accurate within machine precision, valid throughout the entire range of the function, efficient, and readily programmable and adaptable for varying precision and computers. Part I of the present paper is devoted to the error, probability, and related functions.

2. General Methods

Methods used in computing functions vary from expansion in series—power, Taylor, asymptotic, etc.; integration of the differential equation; numerical quadrature; evaluation of continued fractions; polynomial or rational approximations; recurrence techniques; iteration methods; stored tables with the use of interpolation; addition or duplication formulas, etc. Numerical quadrature and integration of the differential equation, while ideal for preparing a table, are generally inefficient for obtaining discrete values. Since the coefficients in approximating polynomials vary with precision, this method will be avoided if possible. The methods chosen for computing the functions will be those which avoid some of the common pitfalls or those pitfalls not easily remedied.

3. Common Pitfalls and Their Avoidance

Computation with alternating series leads to a loss of significant figures due to either or both subtraction and accumulated round-off errors. The use of Taylor series leads to difficulties of premature truncation if any of the derivatives possess zeros. The use of slowly convergent series leads to the difficulty of a heavy accumulation of round-off errors as well as of devising a satisfactory cut-off procedure. In computing with such series one might impose a condition on

[1] Figures in brackets indicate the literature references on p. 212.

the number of terms to be used in the computation. However, this number varies with precision and the method would impose a timing penalty in certain ranges. After the nth term has satisfied the convergence criteria, the sum of the nth to the $(n+p)$th term should be checked for its contribution. The loss of significant figures away from or at the zeros of the functions will be noted by comparing the largest term with the sum and the convergent term. This method of curtailing the series will lead to time savings and frequently avoid overflow and underflow conditions. In using asymptotic expansions the test for convergence should precede the test for divergence.

In computing with continued fractions [2] to avoid premature truncation and or possible overflow or underflow difficulties, the convergents should only be obtained after the numerator and denominator of the ratio are tested for zeros. In this circumstance, the test for convergence should likewise be omitted. In continued fractions where the partial numerators and denominators are always positive, no zeros arise and the value of the continued fraction always lies between two consecutive convergents, except possibly for round-off errors in the calculation. Use of the transformed continued fraction containing only the even or odd convergents of the original fraction, while more efficient generally, may lead in certain cases to a loss of significant figures and to overflow conditions more quickly.

In recurrence techniques, there may be a loss of significant figures or a heavy buildup of round-off errors. Forward and backward techniques or a combination of both may be useful [4, 5]. Attention must be paid to the proper choice of normalization factor, avoiding the use of values at or near zeros.

In using iterative methods, common pitfalls are the imposition of too stringent a convergence criteria and/or a failure to impose a limit on the maximum number of iterations.

4. General Procedures

In addition to detailing the methods of computing, the ASA FORTRAN programs will be presented, prepared in such a way that only a minimal number of changes need be made to accommodate the change in precision. Initially the test ("driver") program will also be given as well as test results and timing in the hope of detecting software and hardware errors.

The authors, hoping to satisfy the needs of all users, earnestly solicit suggestions for improvements and comments relative to testing on other computers. In this way it is hoped that an excellent library of special functions will be established.

5. References

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Automatic Computing Methods for Special Functions

Part I. Error, Probability, and Related Functions

1. Definitions

Error Function

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \qquad \operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

$$= 1 - \operatorname{erf} x$$

$$\operatorname{erf}(-x) = -\operatorname{erf} x \qquad \operatorname{erfc}(-x) = 2 - \operatorname{erfc} x$$

Normal or Gaussian Probability Function

$$Z(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad A(x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} e^{-t^2/2} dt = \int_{-x}^{x} Z(t) dt$$

$$= \operatorname{erf}(x/\sqrt{2})$$

$$P(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \qquad Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt$$

$$= \int_{-\infty}^{x} Z(t) dt \qquad = \int_x^\infty Z(t) dt$$

$$= \tfrac{1}{2}[1 + \operatorname{erf}(x/\sqrt{2})] \qquad = 1 - P(x) = \tfrac{1}{2} \operatorname{erfc}(x/\sqrt{2})$$

Confluent Hypergeometric Function

$$M(\tfrac{1}{2}, \tfrac{3}{2}, -x^2) = (\sqrt{\pi}/2x) \operatorname{erf} x$$

$$M(1, \tfrac{3}{2}, x^2) = (\sqrt{\pi}/2x) e^{x^2} \operatorname{erf} x$$

$$U(\tfrac{1}{2}, \tfrac{1}{2}, x^2) = \sqrt{\pi} e^{x^2} \operatorname{erfc} x$$

Incomplete Gamma Function

$$\gamma(\tfrac{1}{2}, x^2) = 2 \int_0^x e^{-t^2} dt = \sqrt{\pi} \operatorname{erf} x$$

$$\Gamma(\tfrac{1}{2}, x^2) = 2 \int_x^\infty e^{-t^2} dt = \sqrt{\pi} \operatorname{erfc} x$$

2. Method

x, a real variable

NBC = the number of binary digits in the characteristic of a floating point number
NBM = accuracy desired or the maximum number of binary digits in the mantissa of a floating point number
For $x = 0$, $\operatorname{erf} x = 0$; $\operatorname{erfc} x = 1.0$.

The *power series* is used for $|x| \leq \text{ULPS}(=1)$, the upper limit for the power series

$$\text{erf } x = \frac{2x}{\sqrt{\pi}} e^{-x^2} \sum_{N=0}^{\infty} \frac{(2x^2)^N}{1 \cdot 3 \cdot \ldots \cdot (2N+1)} = \frac{2x}{\sqrt{\pi}} e^{-x^2} \left[\sum_{N=1}^{M} T_N + 1 \right]$$

$$\text{where } T_0 = 1,\ D_0 = 1,\ T_N = \frac{2x^2}{2N+1} T_{N-1} = \frac{\text{PWR}}{D_N} T_{N-1};\ D_N = D_{N-1} + 2;\ N \geq 1.$$

Since $\min \sum_{0}^{\infty} T_N = 1$, it is valid to obtain M for the final term by testing for

$$T_M \leq \text{TOLER } (=2^{-\text{NBM}}).$$

NOTE: In the program, in order to minimize round-off errors, summing starts at $N = 1$.

One then obtains $\text{erfc } x = 1 - \text{erf } x$, $\text{erf } (-x) = -\text{erf } x$, $\text{erfc } (-x) = 2 - \text{erfc } x$.

The *continued fraction*

$$\text{erfc } x = \frac{e^{-x^2}}{\sqrt{\pi}} \left[\frac{2x}{2x^2+1} - \frac{1 \cdot 2}{2x^2+5} - \frac{3 \cdot 4}{2x^2+9} - \ldots \right] = \frac{e^{-x^2}}{\sqrt{\pi}} F$$

is used for $1 < |x| \leq \text{ULCF } (=0.83[2^{(\text{NBC}-1)/2}])$, the upper limit (machine bound) for the continued fraction. The above continued fraction yields only even convergents, forming a monotonically increasing sequence, of the following continued fraction

$$\text{erfc } x = \frac{e^{-x^2}}{\sqrt{\pi}} \left[\frac{1}{x} + \frac{1/2}{x} + \frac{1}{x} + \frac{3/2}{x} + \ldots \right].$$

Setting

$F_{-1} = 0$, $F_0 = 2x$, $G_{-1} = 1$, $G_0 = 2x^2 + 1$, $W_1 = 1$, $B_1 = G_0 + 4$ with $W_N = W_{N-1} + 2$, $B_N = B_{N-1} + 4$, $N > 1$, one obtains

$$A_N = -W_N(W_N + 1),\ F_N = A_N F_{N-2} + B_N F_{N-1},\ G_N = A_N G_{N-2} + B_N G_{N-1},\ N \geq 1.$$

If $\left| 1 - \dfrac{F_N/G_N}{F_{N-1}/G_{N-1}} \right| \leq \text{TOLER}(=2^{-\text{NBM}})$, then $F_N/G_N = F$. If F_{N-1}/G_{N-1} denoted by $\text{PREV} \leq F_N/G_N$ the sequence is permitted to continue after first checking for boundedness ($G_N <$ scaling factor, $\text{SCF} = 2^{C2-\text{NBC}}$ where $C2 = 2^{(\text{NBC}-1)}$). If $\text{PREV} > F_N/G_N$, $\text{PREV} = F$. One then obtains the quantities

$$\text{erf } x = 1 - \text{erfc } x,\ \text{erf } (-x) = -\text{erf } x,\ \text{erfc } (-x) = 2 - \text{erfc } x.$$

For $|x| > 0.83\ [2^{(\text{NBC}-1)/2}]$, $\text{erf } |x| = 1$, $\text{erfc } |x| = 0$, the limiting values of the functions.

Beyond $|x| = 1$, a loss of significant figures will occur in subtraction in obtaining erfc x, and in addition round-off errors begin to accumulate more rapidly. If only erf x is needed, a time saving is achieved by extending the power series to $|x| = 2$. The continued fraction should then be used until the appropriate critical value is reached.

While the continued fraction is convergent for $x > 0$, the number of convergents needed increases rapidly and without scaling underflow or overflow occur for x nearing zero. For NBC = 8, NBM = 27, underflow occurred near $x = 0.91$. The partial numerators and denominators F_{N-1},

G_{N-1}, F_{N-2} and G_{N-2} are scaled but greater time saving is achieved by using the power series. If one uses the continued fraction for $|x| < 0.61$ both F_N and G_N must be tested for boundedness.

The maximum precision attainable with the use of the asymptotic expansion

$$\operatorname{erfc} x = \frac{e^{-x^2}}{\sqrt{\pi}x}\left(1+\sum_{N=1}^{\infty}\frac{(-1)^N 1\cdot 3\ \ldots\ (2N-1)}{(2x^2)^N}\right)$$

is for

$x = 4.6$, $|\epsilon| \leq 1$ unit in the 9th significant figure,

$x = 6.7$, $|\epsilon| \leq 1$ unit in the 19th significant figure.

The following table gives an indication of the number of terms needed for particular values of NBM

	Maximum number of terms	
	NBM = 27	NBM = 60
Power series – erf 1	13	21
Continued fraction – erfc 1	24	124
Asymptotic expansion – erfc 6.7	9	47
erfc 9.3	6	17
Continued fraction – erfc 6.7	3	8
erfc 9.3	3	6

3. Range

The range covered is limited by the machine bound for the exponential function.

For $|x| \leq 0.83[2^{(NBC-1)/2}]$, the computations are performed with $|x| \leq 9.3$ for NBC = 8, $|x| \leq 26.5$ for NBC = 11.

For $|x| > .83[2^{(NBC-1)/2}]$, erf $|x| = 1$, erfc $|x| = 0$.

4. Accuracy

At least (NBM−I−3) significant binary digits where I is the number of binary digits representing the integer part of x^2. (This is essentially the accuracy of the exponential routine, with round-off error allowance.)

5. Precision

Variable – by setting the desired NBM.

6. TIMING (Seconds – Univac 1108)

For the range		NBM = 27	NBM = 60	Method
$x = 0(.005)1$	(201 values)	0.08	0.22	Power series
$x = 1(.05)9.4$	(169 values)	.16	.53	Continued fraction
$x = 9.2(.2)26.6$	(88 values)	.....	.13	
Maximum Time/Evaluation (Seconds)		.003	.015	

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7. Testing

To avoid errors due to incorrect arguments, the beginning argument, the interval, and end argument were read in as scaled integers. The generation was performed with integer values, double precision conversion made, then single precision values obtained.

The double precision results were checked, when possible, against those obtained by using the polynomial or rational approximations of Clenshaw [1],[1] Cody [2], Hart [3], and Strecok [6]. In addition, special values were checked by the asymptotic expansion and numerical integration. The single precision results were then checked against the double precision results. In all cases the results obtained agreed within the reported accuracy.

For those desiring both functions accurately with limited precision throughout the range, the approximations of Cody were found the most efficient and accurate.

8. Critical Values

$$\text{erfc } 9.3 = 0.16532\ 44184\ 030157\ (-38)\ \text{(NBC} = 8)$$

$$\text{erfc } 26.5 = 0.22109\ 07664\ 26373\ (-306)\ \text{(NBC} = 11)$$

$$\left.\begin{array}{ll} \text{erf } 4.33 & \\ \text{erf } 4.82 & \\ \text{erf } 5.68 & \\ \text{erf } 6.43 & \end{array}\right\} = 1 - \epsilon \qquad \begin{array}{ll} |\epsilon| < 1(-9) & \text{NBM} = 27 \\ |\epsilon| < 1(-11) & \text{NBM} = 36 \\ |\epsilon| < 1(-15) & \text{NBM} = 48 \\ |\epsilon| < 1(-19) & \text{NBM} = 60 \end{array}$$

9. Special Constants

$$2(\pi)^{-\frac{1}{2}} = 1.12837\ 91670\ 95512\ 57389\ 61589\ 03121$$

$$(\pi)^{\frac{1}{2}}/2 = 0.88622\ 69254\ 52758\ 01364\ 90837\ 41670$$

$$(\pi)^{-\frac{1}{2}} = 0.56418\ 95835\ 47756\ 28694\ 80794\ 51560$$

$$\pi^{\frac{1}{2}} = 1.77245\ 38509\ 05516\ 02729\ 81674\ 83341$$

$$(2\pi)^{-\frac{1}{2}} = 0.39894\ 22804\ 01432\ 67793\ 99460\ 59934$$

$$(2\pi)^{\frac{1}{2}} = 2.50662\ 82746\ 31000\ 50241\ 57652\ 84811$$

$$\pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279$$

$$(2)^{-\frac{1}{2}} = 0.70710\ 67811\ 86547\ 52440\ 08443\ 62104$$

$$2^{\frac{1}{2}} = 1.41421\ 35623\ 73095\ 04880\ 16887\ 24209$$

$$\text{erf } 1 = 0.84270\ 07929\ 49714\ 86934\ 12206$$

$$\text{erf } \rho = 0.5$$

$$\rho = 0.47693\ 62762\ 04469\ 87338$$

[1] Figures in brackets indicate the literature references on p. 217.

$2^{-27} = 0.74505\ \ 80596\ \ 92382\ \ 8125\ (-8)$

$2^{-36} = 0.14551\ \ 91522\ \ 83668\ \ 51806\ \ 64062\ \ 5\ \ (-10)$

$2^{-48} = 0.35527\ \ 13678\ \ 80050\ \ 09293\ \ 55621\ \ 33789\ \ 0625\ \ (-14)$

$2^{-60} = 0.86736\ \ 17379\ \ 88403\ \ 54720\ \ 59622\ \ 40695\ \ 95336\ \ 91406\ \ 25\ \ (-18)$

10. References

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```
1*              SUBROUTINE ERRINT (X, ERF, ERFC)
2*    C               (ASA FORTRAN)
3*    C  USAGE.       CALL ERRINT (X, ERF, ERFC)
4*    C               ENTERING WITH X(A REAL VARIABLE), ONE OBTAINS BOTH ERF X
5*    C               AND ERFC X TO THE MAXIMUM MACHINE ACCURACY POSSIBLE.
6*    C
7*    C               THE PROGRAM, AS SET UP BELOW
8*    C               (1) WITH THE DOUBLE PRECISION TYPE STATEMENT,
9*    C               (2) NBC=11, NBM=60 ON THE FIRST DATA CARD AND
10*   C               (3) THE REMAINING CONSTANTS ON THE SECOND AND THIRD DATA
11*   C               CARDS GIVEN AS DOUBLE PRECISION CONSTANTS, YIELDS DOUBLE
12*   C               PRECISION RESULTS ON THE UNIVAC 1108 SYSTEM.
13*   C
14*   C               SINGLE PRECISION RESULTS ARE OBTAINED BY
15*   C               (1) DELETING THE TYPE STATEMENT
16*   C               (2) SETTING NBC=8, NBM=27 ON THE FIRST DATA CARD
17*   C               (3) CHANGING THE D'S TO E'S ON THE SECOND AND THIRD DATA
18*   C               CARDS.
19*   C               (4) CHANGING FUNCTION TYPE-DABS TO ABS, DEXP TO EXP
20*   C               FOR OTHER VALUES OF NBM, THE CONSTANT ON THE THIRD DATA
21*   C               CARD SHOULD BE GIVEN TO THE CORRESPONDING PRECISION.
22*   C
23*   C         CAUTION.      THE SUBROUTINE CANNOT READILY BE ADAPTED TO
24*   C                       COMPUTE THE ERROR FUNCTION FOR A COMPLEX
25*   C                       ARGUMENT.
26*   C     ERF(X)=(2/SQRT(PI))*INTEGRAL FROM 0 TO X OF EXP (—T**2) DT
27*   C     ERFC(X)=(2/SQRT(PI))*INTEGRAL FROM X TO INFINITY OF
28*   C             EXP (—T**2) DT=1—ERF (X)
29*   C
30*   C     ERF(-X) =-ERF(X)
31*   C     ERFC(—X)=2—ERFC(X)
32*   C
33*   C     NBC=NO. OF BINARY DIGITS IN THE CHARACTERISTIC OF A FLOATING POINT
34*   C        NUMBER.
35*   C     NBM=ACCURACY DESIRED OR MAX. NO. OF BINARY DIGITS IN THE MANTISSA OF
36*   C        A FLOATING POINT NUMBER.
37*   C
38*   C  METHOD.        POWER SERIES FOR ABS(X) .LE. 1,ULPS, UPPER LIMIT FOR
39*   C                 SERIES, CONTINUED FRACTION FOR ABS(X) . GT. 1 AND .LE.
40*   C                 ULCF (UPPER LIMIT OF CONTINUED FRACTION)
41*   C
42*   C  RANGE.         ABS(X) .LE. ULCF        ULCF=.83*(2.**((NBC-1)/2))
43*   C                 ABS(ULCF) APPROX.  9.3, NBC=8
44*   C                                   26.5, NBC=11
45*   C                 BEYOND THIS RANGE THE LIMITING VALUES OF THE FUNCTIONS
46*   C                 ARE SUPPLIED  ERF(INF)=1.0,      ERFC(INF)=0.
47*   C
48*   C  ACCURACY.      ROUTINE WILL RETURN (NBM-I-3) SIGNIFICANT BINARY DIGITS
49*   C                 WHERE I IS THE NUMBER OF BINARY DIGITS REPRESENTING THE
```

```
C                 INTEGER PART OF X**2. (THIS IS ESSENTIALLY THE ACCURACY
C                 OF THE EXPONENTIAL ROUTINE.)
C
C  PRECISION.     VARIABLE-BY SETTING DESIRED NBM.
C
C  MAXIMUM TIME.                    UNIVAC 1108-EXEC II
C     (SECONDS)                        NBM=27       NBM=60
C                                       .003         .015
C
C
C
C
      DOUBLE PRECISION AN, BN, CONS, C1, DN, ERF, ERFC, F, FN, FNM1,
     1     FNM2, FOUR, GN,GNM1  GNM2,ONE, PREV, PWR, RNBC, SCF, SUM,
     2     TN, TOLER, TRRTPI, TWO, ULCF, ULPS, WN, X, Y, YSQ
C
      DATA NBC, NBM/11, 60/
      DATA ONE, TWO, FOUR, ULPS, CONS / 1.D0,2.D0,4.D0,1.D0, .83D0/
      DATA TRRTPI/1.12837916709551257 4D0/
C
      RNBC=NBC
      TOLER=TWO ** (-NBM)
C
C                       TEST ON ZERO
C
      IF (X) 2,1,2
C
    1 ERF=0
      ERFC=ONE
C
      RETURN
C
    2 Y=DABS(X)
      YSQ=Y ** 2
C
      IF (Y-ULPS) 3,3,4
C
C                       MAXIMUM ARGUMENT
C
    4 C1=TWO ** ((RNBC-ONE)/TWO)
      ULCF=CONS * C1
C
C                       SCALE FACTOR
C
      SCF=TWO ** (C1 ** 2-RNBC)
C
C                       LIMITING VALUE
C
      IF (Y-ULCF) 10,10,11
C
   11 ERF=ONE
      ERFC=0
```

```
      GO TO 7
C
C                METHOD——POWER SERIES
C
   3  SUM=0
      DN=ONE
      TN=ONE
      PWR=TWO*YSQ
   6  DN=DN + TWO
      TN=PWR*TN/DN
      SUM=TN + SUM
C
C                        TOLERANCE CHECK
C
      IF (TN-TOLER) 5,6,6
C
   5  ERF=(SUM + ONE) *TRRTPI*Y*DEXP(—YSQ)
      ERFC=ONE — ERF
C
C                          NEGATIVE ARGUMENT
C
   7  IF (X) 8, 9, 9
C
   8  ERF=—ERF
      ERFC=TWO—ERFC
C
   9  RETURN
C
C                METHOD——CONTINUED FRACTION
C
  10  FNM2=0
      GNM2=ONE
      FNM1=TWO * Y
      GNM1=TWO * YSQ + ONE
C
      PREV=FNM1/GNM1
C
      WN=ONE
      BN=GNM1 + FOUR
C
  14  AN=—WN * (WN + ONE)
C
      FN=BN * FNM1 + AN * FNM2
      GN=BN * GNM1 + AN * GNM2
C
      F=FN/GN
C
C                        TOLERANCE CHECK
C
      IF (DABS (ONE- (F/PREV))—TOLER) 12, 12, 13
  13  IF (PREV—F) 17, 17, 18
C       BOTH FN AND GN MUST BE TESTED IF ABS (X) .LT. .61
```

```
154*       17     IF (GN .LT. SCF) GO TO 16
155*    C
156*    C                          SCALING
157*    C
158*       15     FN=FN/SCF
159*              GN=GN/SCF
160*              FNM1=FNM1/SCF
161*              GNM1=GNM1/SCF
162*    C
163*       16     FNM2=FNM1
164*              GNM2=GNM1
165*              FNM1=FN
166*              GNM1=GN
167*    C
168*              WN=WN+TWO
169*              BN=BN+FOUR
170*              PREV=F
171*              GO TO 14
172*    C
173*       18     F=PREV
174*       12     ERFC=F*DEXP (-YSQ)*TRRTPI/TWO
175*              ERF=ONE-ERFC
176*    C
177*              GO TO 7
178*    C
179*              END
```

```
C
C
C   TEST PROGRAM FOR DOUBLE PRECISION ERROR FUNCTION AND COMPLI-
C     MENTARY ERROR FUNCTION      (UNIVAC 1108)
C                                 CALL ERRINT (X, ERF, ERFC)
C
C
      DOUBLE PRECISION DN, DSCALE, ERF, ERFC, X
C
      DIMENSION M(12)
      DIMENSION X (500), ERF (500), ERFC (500)
C
      DATA M(1), M(2), M(3), M(4), M(5), M(6), M(7), M(8), M(9),
     1  M(10), M(11), M(12)/0, 5, 100, 100, 10, 5, 95, 10,
     2       95, 5, 270, 10/
      J=1
C
    1 N=M(J)
      NINCR=M(J+1)
      NENDX=M(J+2)
      DSCALE=M(J+3)
C
      I=1
    5 DN=N
      X(I)=DN/DSCALE
C
      IF (N-NENDX) 2, 3, 3
C
    2 I=I+1
      N=N+NINCR
      GO TO 5
C
    3 DO 4 L=1, I
    4 CALL ERRINT (X (L), ERF (L), ERFC (L))
C
      WRITE (6, 1002)
      DO 6 L=1, I
    6 WRITE (6, 1001) X (L), ERF (L), ERFC (L)
C
      IF (J .EQ. 9) STOP
      J=J+4
      GO TO 1
C
 1001 FORMAT (3D26. 18)
 1002 FORMAT (1H1, 11X, 1HX, 23X, 6HERF(X), 20X, 7HERFC(X)./)
C
      END
```

X	ERF(X)	ERFC(X)
.00000000000000000	.00000000000000000	.10000000000000000 + 001
.50000000000000000 − 001	.56371977970166236 − 001	.94362822022983377 + 000
.10000000000000000 + 000	.11246291601828489 2 + 000	.88753708398171510 9 + 000
.15000000000000000 + 000	.16795957142736349 5 + 000	.83200402857263650 6 + 000
.20000000000000000 + 000	.22270258921047845 4 + 000	.77729741078952154 6 + 000
.25000000000000000 + 000	.27632639016823693 5 + 000	.72367360983176306 5 + 000
.30000000000000000 + 000	.32862675945912742 9 + 000	.67137324054087257 2 + 000
.35000000000000000 + 000	.37938205356231032 2 + 000	.62061794643768967 8 + 000
.40000000000000000 + 000	.42839235504666845 5 + 000	.57160764495333154 5 + 000
.45000000000000000 + 000	.47548171978692367 7 + 000	.52451828021307632 4 + 000
.50000000000000000 + 000	.52049987781304654 1 + 000	.47950012218695345 9 + 000
.54999999999999999 + 000	.56323236632510895 7 + 000	.43667663367489104 4 + 000
.59999999999999999 + 000	.60385609084792592 4 + 000	.39614390915207407 6 + 000
.65000000000000000 + 000	.64202932735567184 3 + 000	.35797067264432815 7 + 000
.70000000000000000 + 000	.67780119383741847 6 + 000	.32219880616258152 4 + 000
.75000000000000000 + 000	.71115563365351513 3 + 000	.28884436634648486 7 + 000
.79999999999999999 + 000	.74210096470766048 5 + 000	.25789035292339516 + 000
.84999999999999999 + 000	.77066805760835253 0 + 000	.22933194239164747 0 + 000
.90000000000000000 + 000	.79690821242283213 1 + 000	.20309178577167870 + 000
.95000000000000000 + 000	.82089080727327794 4 + 000	.17910919272672205 6 + 000
.10000000000000000 + 001	.84270079294971487 1 + 000	.15729920705028513 0 + 000

X	ERF(X)	ERFC(X)
.10000000000000000 + 001	.84270079294971487 1 + 000	.15729920705028513 0 + 000
.15000000000000000 + 001	.96610514647531072 8 + 000	.33894853524689273 5 − 001
.20000000000000000 + 001	.99532265018952735 + 000	.46777349810472658 2 − 002
.25000000000000000 + 001	.99959304798255504 3 + 000	.40695201744495893 8 − 003
.30000000000000000 + 001	.99997790950300141 5 + 000	.22090496998585442 0 − 004
.35000000000000000 + 001	.99999256901627659 + 000	.74309837234141277 4 − 006
.40000000000000000 + 001	.99999984582742101 + 000	.15417257900280019 5 − 007
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.65000000000000000 + 001	.10000000000000000 + 001	.38421483271206478 3 − 019
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.75000000000000000 + 001	.10000000000000000 + 001	.27766493860305693 9 − 025
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.84999999999999999 + 001	.10000000000000000 + 001	.27623240713337719 3 − 032
.90000000000000000 + 001	.10000000000000000 + 001	.41370317465138108 1 − 036
.95000000000000000 + 001	.10000000000000000 + 001	.37692144856548809 9 − 040

X	ERF(X)	ERFC(X)
.95000000000000000 + 001	.10000000000000000 + 001	.376921448565488099 − 040
.10000000000000000 + 002	.10000000000000000 + 001	.208848758376254514 − 044
.10500000000000000 + 002	.10000000000000000 + 001	.703592809017752444 − 049
.11000000000000000 + 002	.10000000000000000 + 001	.144086613794369515 − 053
.11500000000000000 + 002	.10000000000000000 + 001	.179330964357678251 − 058
.12000000000000000 + 002	.10000000000000000 + 001	.135626116920590467 − 063
.12500000000000000 + 002	.10000000000000000 + 001	.623194278197991275 − 069
.13000000000000000 + 002	.10000000000000000 + 001	.173955731546672518 − 074
.13500000000000000 + 002	.10000000000000000 + 001	.294943311325798927 − 080
.14000000000000000 + 002	.10000000000000000 + 001	.303722984775031350 − 086
.14500000000000000 + 002	.10000000000000000 + 001	.189939594197950386 − 092
.15000000000000000 + 002	.10000000000000000 + 001	.721299417245120960 − 099
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.22000000000000000 + 002	.10000000000000000 + 001	.162190586093347461 − 211
.22500000000000000 + 002	.10000000000000000 + 001	.344534886046460488 − 221
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.24000000000000000 + 002	.10000000000000000 + 001	.164898258315193574 − 251
.24500000000000000 + 002	.10000000000000000 + 001	.474936126406738376 − 262
.25000000000000000 + 002	.10000000000000000 + 001	.830017257119653210 − 273
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.26000000000000000 + 002	.10000000000000000 + 001	.566319240885615157 − 295
.26499999999999999 + 002	.10000000000000000 + 001	.221090766426373696 − 306
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(Paper 74B3–331)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
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Publications of the National Bureau of Standards *

Selected Abstracts

Haber, S., **Stochastic quadrature formulas,** *Math. Computation* **23,** *No. 108, 751–764 (Oct. 1969).*

Key words: Degree of precision; Hadamard matrices; integration; Monte Carlo; multiple integrals; numerical integration; numerical methods; quadrature; random.

A class of formulas for the numerical evaluation of multiple integrals is described, which combines features of the Monte Carlo and the classical methods. For certain classes of functions—defined by smoothness conditions—these formulas provide the fastest possible rate of convergence to the integral. Asymptotic error estimates are derived, and a method is described for obtaining good a posteriori error bounds when using these formulas. Equal-coefficients formulas of this class, of degrees up to 3, are constructed.

Haber, S., Osgood, C. F., **On the sum $\Sigma\langle n\alpha\rangle^{-t}$ and numerical integration,** *Pacific J. Math.* **31,** *No. 2, 383–394 (1969).*

Key words: Continued fraction; diophantine approximation; integration; multiple integration; number theory; numerical analysis; numerical integration; periodic functions, quadrature.
It is shown that a certain problem in Diophantine Approximation the estimation of sums of the form $\sum_{n=k}^{A} \langle n\alpha\rangle^{-t}$, where α is an irrational number, "$\langle x \rangle$" denotes the distance from x to the nearest integer, and A is a fixed number greater than 1—is realted to the estimation of the error in Korobov's method of numerical integration of periodic functions of several real variables. Some estimates of such sums are found, for certain classes of irrational numbers.

Wampler, R. H., **An evaluation of linear least squares computer programs: A summary report** *(Proc. 14th Conf. Design of Experiments in Army Research Development and Testing, Edgewood Arsenal, Md., October 23, 1968), ARO–D Report 69–2, 103–126 (U.S. Army Research Office-Durham, Durham, North Carolina, Sept. 1969).*

Key words: Computer programs; curve fitting; Gram-Schmidt orthogonalization; Householder transformations; iterative refinement; least squares; linear equations; orthogonal polynomials; orthogonalization regression; rounding error; stepwise regression.

Two linear least squares test problems, based on fifth degree polynomials, have been run on more than twenty different computer programs in order to assess their numerical accuracy. Among the programs tested were representatives from various statistical packages as well as some from the SHARE library. Essentially four different algorithms were used in the various programs to obtain the coefficients of the least squares fits. The tests were run on several different computers, in double precision as well as single precision. By comparing the coefficients reported, it was found that those programs using orthogonal Householder transformations or Gram-Schmidt orthonormalization were much more accurate than those using elimination algorithms. Programs using orthogonal polynomials (suitable only for polynomial fits) also proved to be superior to those using elimination algorithms. The most successful programs accumulated inner products in double precision and made use of iterative refinement procedures. In a number of programs, the coefficients reported in one test problem were sometimes completely erroneous, containing not even one correct significant digit.

Other NBS Publications

J. Res. Nat. Bur. Stand. (U.S.), 74A (Phys. and Chem.), No. 3 (May–June 1970), SD Catalog No. C13.22/sec.A:74/3, $1.25.

The crystal structure of hydrazinium trinitromethide $(N_2H_5^+[C(NO_2)_3]^-$ at ~ −160 °C. B. Dickens.
A refinement of the crystal structure of $Na_2CO_3 \cdot H_2O$. B. Dickens, F. A. Mauer, and W. E. Brown.
Anisotropic reaction kinetics of oxygen with pyrolytic graphite. W. S. Horton.
The viscosity and thermal conductivity of dilute gaseous hydrogen from 15 to 5000 K. H. J. M. Hanley, R. D. McCarty, and H. Intemann.
Temperature classification of the spectra of dysprosium (Dy I, Dy II). A. S. King, J. G. Conway, E. F. Worden, and C. E. Moore.

Papers presented at the Conference, Electronic Density of States, National Bureau of Standards, Gaithersburg, Maryland November 3–6, 1969

Optical density of states ultraviolet photoelectric spectroscopy. W. E. Spicer.
Beyond the one-electron approximation: Density of states for interacting electrons. L. Hedin, B. Lundqvist, and S. Lundqvist.
Ion-neutralization spectroscopy. H. D. Hagstrom.
Local theory of disordered systems. W. H. Butler and W. Kohn.
Density of states information from low temperature specific heat measurements. P. A. Beck and H. Claus.

J. Res. Nat. Bur Stand (U.S.), 74A (Phys. and Chem.), No. 4(July–Aug. 1970), SD Catalog No. C13.22/sec.A: 74/4, $2.00
Solubility of $CaHPO_4 \cdot 2H_2O$ in the system $Ca(OH)_2-H_3PO_4-H_2O$ at 5, 15, 25, and 37.5 °C. T. M. Gregory, E. C. Moreno, and W. E. Brown.
Effect of oxide additions on the polymorphism of tantalum pentoxide II. "Stabilization" of the high temperature structure type. R. S. Roth, J. L. Waring, and W. S. Brower.
Effect of oxide additions on the polymorphism of tantalum pentoxide III. "Stabilization" of the low temperature structure type. R. S. Roth and J. L. Waring.
The dissociation constant of *m*-nitroanilinium ion in water-tetrahydrofuran solvents at 25 °C. R. A. Robinson.
Specific heats of fluorine at coexistence. R. D. Goodwin and R. Prydz.
A theoretical investigation of the configurations $(3d+4s)^6 4p$ in neutral manganese (Mn I). C. Roth.
A literature survey of the chemistry of flame inhibition. E. C. Creitz.
The NBS Alloy Data Center: Description of Index to the literature. G. C. Carter, D. J. Kahan, L. H. Bennett, J. R. Cuthill, and R. C. Dobbyn.

Papers presented at the Conference, Electronic Density of States, National Bureau of Standards, Gaithersburg, Maryland, November 3–6, 1969

What is a quasi-particle? J. R. Schrieffer.
Electronic densities of states from x-ray photoelectron spectroscopy. C. S. Fadley and D. A. Shirley.
Tunneling measurements of superconducting quasi-particle density of states and calculation of phonon spectra. J. M. Rowell.
Pauli paramagnetism in metals with high densities of states. S. Foner.

Relevance of Knight shift measurements to the electronic density of states. L. H. Bennett, R. E. Watson, and G. C. Carter.

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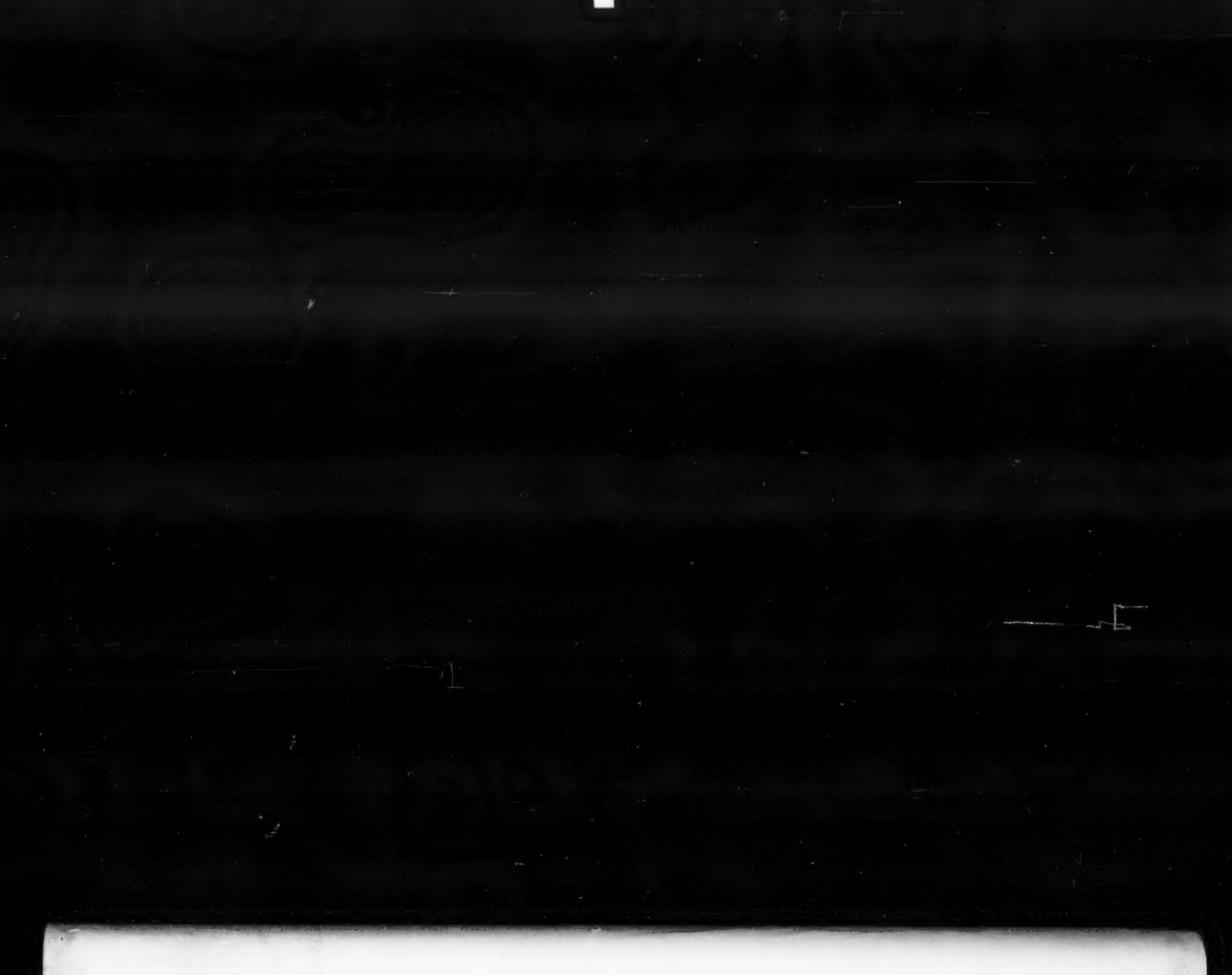
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